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ON WAVE-INDUCED STRESS IN A SHIP EXECUTING SYMMETRIC MOTIONS

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CONTENTS

The equation of motion of a simple beam in head waves is solved in terms of modal responses. Examination of the resulting expression for wave-induced bending moment indicates that at lower wave frequencies large fluctuating stresses are generally associated with 'ship–wave matching', a phenomenon governed by the relative geometry of ship and wave; whereas large stresses in the higher frequency range are the result of 'resonant encounter', during which the encounter frequency of ship with wave corresponds to a natural vibration frequency of the ship as a beam. The contrasting characteristics of these different response mechanisms are shown to provide a rational explanation of the fluctuating stresses induced in large or flexible ships in confused seas.

1. Introduction

'There seems little doubt that, for design purposes, ship research must in the future be increasingly concerned with the dynamic nature of the loading and response of the structure.' This comment concludes the Introduction of the Report to the Fourth International Ship Structures Congress by the Committee concerned with Design Procedure (Committee 10), and it is typical of many such remarks recorded in the *Proceedings* of that Congress in 1970. The work described here is concerned with these problems. It is intended to be a first step in the development of a unified approach to ship structural dynamics, bridging where possible the gap between hydrodynamics and structural analysis, and drawing together within a common framework the different calculations commonly made for the various types of excitation and response. The approach is based on the belief that a unified theory is a prerequisite to achieving satisfactory methods for rational ship design. The criteria checked at the design stage must embody both strength and stiffness considerations.

Problems of propeller and wave-excited vibrations have recently been accentuated by the general growth in ship size and resulting reduction in natural frequencies of primary and secondary structures. The increased power and speed and the decreased structural damping of many modern ships have also contributed to these problems. Such matters should be considered at the design stage, and analysis of a given design should examine these aspects along with the properties commonly associated with conventional seakeeping and wave bending moment calculations. That these should all be analysed by a unified theory is logical. For future specialized ships that may be very large, slender and relatively flexible, any other approach may be invalid.

A fundamental problem is that of obtaining long-term stress predictions.† Two methods may be distinguished: (i) statistical analysis of full-scale stress measurements; (ii) theoretical methods based on model tests or computed response amplitude operators, combined with weather statistics. It is to the latter that attention is particularly drawn here, since these methods seem to hold out the most promise for new types of ship, whose response may not always be satisfactorily extrapolated from previous tried designs. But where possible results should of course be correlated with full-scale measurements, such as from the twelve ships (including tankers, bulk carriers and fast cargo ships) currently subject to extensive instrumentation under the auspices of the British Ship Research Association.

Consider first the behaviour of a ship in waves characterized by its response in heave and pitch. The fundamental phenomenon has been discussed most elegantly in the classic paper by Weinblum & St Denis (1950). Theoretical methods of obtaining response amplitude operators by 'strip theory' and 'slender body theory' have been described by Korvin-Kroukowsky & Jacobs (1957) and Grim (1960). Typical results of computations based on these theories are given by Fukuda (1966), Gerritsma & Beukelman (1967) and Kaplan (1969). Some representative model test results are in the papers by Moor (1967) and Joosen, Wahab & Woortman (1968). Much research in all these areas is recorded in the reports of the relevant committees of the International Ship Structures Congress and the International Towing Tank Conference (e.g. I.S.S.C. (1970) and I.T.T.C. (1969)). It is a feature of most of this work that results are only given for wavelengths greater than about half a ship length. Smaller waves were of little interest since in a seaway they would have had relatively little energy, besides which they are difficult to reproduce in

† The still water zero frequency bending moment is here ignored, since it may be considered independently of dynamic wave effects.

experimental tanks. With the increase in size of today's ships this deficiency is becoming an embarrassment.

Next consider wave-induced vibration. This has only lately attracted the attention of many naval architects, because of the experience of recent large tankers, container ships and Great Lakers. Analyses have been given by Goodman (1971) and van Gunsteren (1970) in which wave-excited stresses have been calculated ignoring heaving and pitching. Full-scale measurements have been given by Cleary, Robertson & Yagle (1971) for Great Lakers, and by Little & Lewis (1971) for tankers and bulk carriers. No comprehensive results of model tests have been reported, as far as these authors are aware.

It may seem surprising that this subject has been treated in the past in a somewhat fragmented manner. Response in waves has certainly always been a matter of fundamental importance in assessing the performance of ships. A ship's very survival depends on the success of the naval architect in ensuring that the stresses at all times remain within certain limits. But almost invariably design stresses have effectively been calculated (if at all) by estimating amidship bending moments on the basis of the ship responding in waves as a rigid beam. Only recently has consideration been given to the flexibility of the beam and analysis made of wave-induced vibrations, but as a separate phenomenon. It may be argued that traditional methods are amply vindicated by the satisfactory performance of most of the ships today sailing the high seas. But it is now recognized by many that these methods are in some cases deficient. At comparatively moderate levels of vibratory stress cumulative damage by fatigue could have a marked influence. And it has been suggested (for example by Committee 3 of the Fourth I.S.S.C. in 1970) that in certain large ships wave vibration stresses could be of comparable magnitudes to normal wave stresses. It is intended that the analysis which follows should lead to a rational assessment of these and other problems of the structural dynamics of ships.

The first stage is to achieve an understanding of the nature of the various problems, before proceeding to the development of computational tools for practical design. It was proposed by Bishop (1971) that this should be achieved by analysing the structural dynamics of the ship in terms of modal responses; the idea was advanced in a public lecture so that only a rudimentary analysis could be given by way of illustration. The underlying theoretical ideas were developed and presented in the technical literature by Bishop, Eatock Taylor & Jackson (1973). This paper records first steps in the process of translating the basic theoretical concepts into something of practical value. It describes a modal analysis of a simple floating beam in symmetric motions, under the limiting assumptions imposed by a severe simplification of the hydrodynamic behaviour. It outlines the development of modes, motion responses and bending-moment response amplitude operators for the beam in a confused seaway.

Notation

a	wave amplitude
A, B	wave spectrum parameters
$b(x)$	beam of ship at point x
b_0	amidship beam of ship
C_n	normalizing constant in directional wave spectrum
$EI(x)$	flexural rigidity of ship at point x
$f_0^{\mathrm{L}}(kl), f_0^{\mathrm{W}}(kl), f_1^{\mathrm{H}}(kl)$	functions for amidship bending moments

Fr	Froude number $(=U/\sqrt{(gl)})$
$F_{\mathrm{H}}(x,t)$	force per unit length acting on ship
$\bar{F}_r(kl,t)$	generalized force corresponding to rth characteristic mode
g	gravitational acceleration
$h_{\frac{1}{3}}$	significant wave height
$h_0^{\mathrm{W}}(\omega,n)$, $h_1(\omega,n)$	amidship bending moment operators for directional seas
$H_1(\omega_{\mathrm{e}})$	frequency response function
k	wavenumber $(=2\pi/\Lambda)$
l	length of ship
$M(x,t)$	bending moment at x
$M_r^{\mathrm{H}}(kl,x,t)$, $M_s^{\mathrm{L}}(kl,x,t)$	generalized bending moments
$M^{\mathrm{W}}(kl,x,t)$	quasi-static bending moment
$\bar{M}_s(x)$	bending moment during oscillations in still water
M_{H}^*, M_{W}^*	root mean square bending moment coefficients
n	directional spread exponent in wave spectrum
$p_r(t)$	rth principal coordinate
r, s	index defining mode $(r=-1,0,1,2,\ldots,s=-1,0)$
$S(\omega)$	unidirectional wave spectrum
$S^*(\omega,\theta)$	directional wave spectrum
$S_1(\omega_{\mathrm{e}},U)$	transformed wave spectrum
$S_1'(\omega_{\mathrm{e}},U)$	averaged, transformed wave spectrum
t	time
U	forward speed of ship
$w(x,t)$	downward deflexion at x
β_r	$=[(\mu'\omega_r^2-\rho gb)/EI]^{\frac{1}{4}}$
$\zeta(x,t)$	wave depression
η	damping factor
θ	angle between wave component and predominant wave direction
Λ	wavelength
$\mu(x)$	mass per unit length of ship at x
$\mu_0(x)$	added mass per unit length of ship at x
$\mu'(x)$	virtual mass per unit length of ship at x
$\bar{\mu}_r'$	generalized virtual mass of ship in rth characteristic mode
ξ	non-dimensional coordinate $(=x/l)$
ρ	water density
σ	non-dimensional parameter $(=\frac{1}{2}kl)$
τ	non-dimensional frequency $(=\omega_0/\omega_1)$
ω_{e}	frequency of encounter
ω_{m}	ship–wave matching frequency
ω_r	rth characteristic frequency

2. Response of a ship in progressive sinusoidal waves

2.1. *The idealization of a simple floating beam*

The simple structural model to which attention is here directed is a floating beam in which shear deformation and rotary inertia are ignored. The beam is initially considered to have non-uniform flexural stiffness $EI(x)$ per unit length and mass distribution $\mu(x)$ per unit length, and subsequently the special case of a uniform beam is developed in some detail. Discussion of the latter will also be found in Bishop *et al.* (1973).

Sweeping simplifications are made concerning the behaviour of the fluid in which the beam floats. We assume that the interaction of beam and fluid may be represented through use of the sectional added mass $\mu_0(x)$, and added stiffness $\rho g b(x)$, per unit length. These quantities are taken to be time independent during arbitrary motions of the beam; thus for harmonic vibrations they do not depend on frequency. We ignore hydrodynamic damping, apart from brief consideration of possible effects of dissipative damping due to wave generation. And we assume that vertical motions of the beam are independent of any forward velocity it may have through the fluid.

When discussing wave excitation, we further assume that the exciting force is simply the hydrostatic resultant due to the wave elevation. Thus we ignore interference between beam and wave, and the orbital velocities of wave particles which lead to attenuation of pressure amplitudes with depth (the Smith effect). The waves are taken to be sinusoidal progressive waves in deep water, with wavenumber k, encountered at frequency ω_e.

We refer to this mathematical model as the simple floating beam.

2.2. *The equation of motion and its solution in terms of the characteristic modes of a simple floating beam*

The equation of vertical motion of the beam is written in terms of the downward deflexion $w(x,t)$. At a section x in this simple idealization the hydrodynamic force opposing the motion is

$$\mu_0 \frac{\partial^2 w}{\partial t^2} + \rho g b w.$$

The wave excitation force is $\rho g b \zeta$, where the wave depression ζ is given by

$$\zeta(x,t) = a \sin(\omega_e t - kx). \tag{2.1}$$

Hence the equation of motion (ignoring damping at this stage) is

$$\mu \frac{\partial^2 w}{\partial t^2} = -\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 w}{\partial x^2}\right) - \mu_0 \frac{\partial^2 w}{\partial t^2} - \rho g b w + \rho g b \zeta,$$

which may be written in the form

$$\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 w}{\partial x^2}\right) + \rho g b w + \mu' \frac{\partial^2 w}{\partial t^2} = \rho g b \zeta, \tag{2.2}$$

where the virtual mass per unit length is

$$\mu' = \mu + \mu_0.$$

Let us first examine the case of free vibrations. We consider the homogeneous equation, and assume that motions of the form

$$w(x,t) = \phi_r(x) \sin \omega_r t \tag{2.3}$$

can take place. The quantities ω_r and $\phi_r(x)$ are the natural frequency and corresponding characteristic mode for the non-uniform beam. The functions $\phi_r(x)$ satisfy

$$\frac{d^2}{dx^2}\left(EI\frac{d^2\phi_r}{dx^2}\right)+\rho g b\phi_r-\mu'\omega_r^2\phi_r=0. \tag{2.4}$$

Hence they may be shown to satisfy the orthogonality relations

$$\int_0^l \mu'(x)\,\phi_r(x)\,\phi_s(x)\,dx=\begin{Bmatrix}0 & \text{if} & r\neq s\\ \bar{\mu}_r' & \text{if} & r=s\end{Bmatrix}, \tag{2.5}$$

where $\bar{\mu}_r'$ is the generalized mass corresponding to the rth characteristic mode.

Thus the beam may vibrate in one or more of an infinity of characteristic modes. The mode shapes $\phi_r(x)$ depend on the distribution of virtual mass and stiffness of the beam; though they are not generally expressible in simple analytical form, they reduce to simple well-known results in the case of a uniform beam. The lowest two modes for a uniform beam are found to have certain special properties, and it is convenient to associate these with $r=-1, 0$. We shall use this notation both for uniform and for non-uniform beams. The higher modes correspond to $r=1, 2, 3, \ldots, \infty$.

To obtain the solution of the equation of motion in waves, we therefore let

$$w(x,t)=\sum_{r=-1}^{\infty}p_r(t)\,\phi_r(x), \tag{2.6}$$

where $p_r(t)$ is the rth principal coordinate of the system. Substituting this in the equation of motion, (2.2), we obtain

$$\sum_{r=-1}^{\infty}p_r(t)\frac{d^2}{dx^2}\left(EI(x)\frac{d^2\phi_r(x)}{dx^2}\right)+\sum_{r=-1}^{\infty}p_r(t)\,\rho g b(x)\,\phi_r(x)+\sum_{r=-1}^{\infty}\ddot{p}_r(t)\,\mu'(x)\,\phi_r(x)$$
$$=a\rho g b(x)\sin(\omega_e t-kx).$$

Using (2.4) for $r=-1, 0, 1, 2, \ldots$, we find that this may be written

$$\sum_{r=-1}^{\infty}\ddot{p}_r(t)\,\mu'(x)\,\phi_r(x)+\sum_{r=-1}^{\infty}\omega_r^2 p_r(t)\,\mu'(x)\,\phi_r(x)=a\rho g b(x)\sin(\omega_e t-kx).$$

Next we multiply this equation by $\phi_s(x)$, integrate over the range $0\leqslant x\leqslant l$, and use the orthogonality relations of (2.5), to write the equation of motion in the form

$$\bar{\mu}_r'(\ddot{p}_r+\omega_r^2 p_r)=a\bar{F}_r(kl,t),$$

where we have defined the rth generalized excitation force as

$$\bar{F}_r(kl,t)=\int_0^l \rho g b(x)\,\phi_r(x)\sin(\omega_e t-kx)\,dx\quad(r=-1,0,1,2,\ldots). \tag{2.7}$$

The steady-state modal responses are given by

$$p_r(t)=\frac{a\bar{F}_r(kl,t)}{\bar{\mu}_r'(\omega_r^2-\omega_e^2)}.$$

Hence the solution of the equation of forced vibration in waves is

$$w(x,t)=\sum_{r=-1}^{\infty}\frac{a\bar{F}_r(kl,t)\,\phi_r(x)}{\bar{\mu}_r'(\omega_r^2-\omega_e^2)}. \tag{2.8}$$

We notice that resonance is possible in any mode r if $\omega_r \to \omega_e$. Moreover, the response in mode r behaves as $\bar{F}_r(kl, t)$, and may be magnified if kl has an appropriate value. Response near the lowest two natural frequencies is equivalent to 'sea-keeping' behaviour, whereas response in the higher modes corresponds to what is conventionally considered ship vibration. For a non-uniform ship there seems to be little theoretical justification for this distinction, but it has proved useful in practice. This is because for most ships, which as beams are relatively 'stiff', the frequencies ω_{-1} and ω_0 are close, and considerably lower than ω_1, ω_2, etc.: the responses are therefore considered independently in the low- and high-frequency régimes.

2.3. *Wave bending moments in the simple beam*

The bending moment at any section x may be found from the moment–curvature relation

$$M(x,t) = EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}. \tag{2.9}$$

If the series form of $w(x,t)$ given in (2.8) is substituted in (2.9), this leads to the result

$$M(x,t) = \sum_{r=-1}^{\infty} \frac{a\bar{F}_r(kl,t)\,EI(x)\,\phi_r''(x)}{\bar{\mu}_r'(\omega_r^2-\omega_e^2)}, \tag{2.10}$$

where
$$\phi_r''(x) = \frac{d^2\phi_r}{dx^2}.$$

From the point of view of calculating the bending moment at some section x, however, this series form of $M(x,t)$ is not satisfactory: the modal contributions converge only very slowly. A more useful form, which also provides a very simple expression for the special case of a rigid beam, may be obtained using some identities derived in appendix A. We first define the quantity

$$\bar{M}_r(x) = EI(x)\,\phi_r''(x).$$

It is shown in appendix A that

$$\bar{M}_r(x) = \int_0^x [\mu'(x_0)\,\omega_r^2 - \rho g b(x_0)]\,\phi_r(x_0)\,(x-x_0)\,dx_0,$$

and we see that this is the amplitude of bending moment at x due to oscillations of unit magnitude in still water, at frequency ω_r in the rth characteristic mode. [It is possible to attach a physical meaning to these still water fluctuating bending moments for the case of a rigid beam. Then only two modes exist, ϕ_{-1} and ϕ_0, since there are only two rigid body degrees of freedom for motions in the vertical plane. These correspond to pitch and heave respectively. Thus $\bar{M}_{-1}(x)$ is the amplitude of bending moment at x due to oscillations of unit amplitude in still water in the pitching mode, at frequency ω_{-1}. And a similar meaning attaches to $\bar{M}_0(x)$ for unit heaving. These bending moments arise because, in contrast to the behaviour for a uniform beam pitching or heaving at the appropriate natural frequency, the buoyancy forces are no longer exactly opposed by the D'Alembert inertia forces at every point along the beam. When the forces are integrated over the length of the beam balance is achieved, but at any section within the non-uniform beam the resulting out of balance forces give rise to a bending moment.]

The next step in finding an alternative expression for the total wave bending moment $M(x,t)$ in (2.10) is to use the second identity derived in appendix A:

$$\mu'(x)\sum_{r=-1}^{\infty}\frac{\bar{F}_r(kl,t)}{\bar{\mu}_r'}\,\phi_r(x) = \rho g b(x)\sin(\omega_e t - kx).$$

Interchanging the order of summation and integration in (2.10) we obtain finally

$$M(x,t) = a\int_0^x \left[\rho g b(x_0)\sin(\omega_e t - kx) + \sum_{r=-1}^{\infty} \frac{[\mu'(x_0)\,\omega_e^2 - \rho g b(x_0)]}{\overline{\mu}_r'(\omega_r^2-\omega_e^2)} \overline{F}_r(kl,t)\,\phi_r(x_0)\right](x-x_0)\,dx_0. \tag{2.11}$$

As might be expected, this alternative form of $M(x,t)$ may also be obtained directly by physical arguments. If the external forces acting on the beam are F_H per unit length, the bending moment at x may be written

$$M(x,t) = \int_0^x \left(F_H - \mu\frac{\partial^2 w}{\partial t^2}\right)(x-x_0)\,dx_0.$$

Now in deriving the equation of motion, we have assumed that the external forces are the fluid forces given by

$$F_H = \rho g b\zeta - \left[\mu_0\frac{\partial^2 w}{\partial t^2} + \rho g b w\right].$$

Hence we have

$$M(x,t) = \int_0^x \left[\rho g b(x_0)\,\{\zeta(x,t) - w(x,t)\} - \mu'(x)\frac{\partial^2 w(x,t)}{\partial t^2}\right](x-x_0)\,dx_0.$$

If the series form of $w(x,t)$ in (2.8) and the wave depression $\zeta(x,t)$ given in (2.1) are substituted into this equation, the expression in (2.11) is again obtained.

In order to distinguish between lower and higher frequency responses, as suggested above, it is convenient to write $M(x,t)$ such that the summation includes only contributions from the higher modes $r = 1, 2, \ldots$. Thus

$$\begin{aligned} M(x,t) &= a\int_0^x \rho g b(x_0)\left\{\sin(\omega_e t - kx) - \left[\frac{\overline{F}_{-1}(kl,t)\,\phi_{-1}(x_0)}{\overline{\mu}'_{-1}\omega_{-1}^2} + \frac{\overline{F}_0(kl,t)\,\phi_0(x_0)}{\overline{\mu}'_0\omega_0^2}\right]\right\}(x-x_0)\,dx_0 \\ &+ a\int_0^x \sum_{r=1}^{\infty}\frac{[\mu'(x_0)\,\omega_e^2 - \rho g b(x_0)]}{\overline{\mu}_r'(\omega_r^2-\omega_e^2)}\overline{F}_r(kl,t)\,\phi_r(x_0)\,(x-x_0)\,dx_0 \\ &+ a\omega_e^2\left[\frac{\overline{F}_{-1}(kl,t)\,\overline{M}_{-1}(x)}{\omega_{-1}^2\overline{\mu}'_{-1}(\omega_{-1}^2-\omega_e^2)} + \frac{\overline{F}_0(kl,t)\,\overline{M}_0(x)}{\omega_0^2\overline{\mu}'_0(\omega_0^2-\omega_e^2)}\right], \end{aligned} \tag{2.12}$$

where we have used the quantities $\overline{M}_{-1}$, $\overline{M}_0$ defined previously. For convenience this can be written in the form

$$M(x,t) = M^W(kl,x,t) + \sum_{r=1}^{\infty}\frac{\omega_e^2-\omega_0^2}{\omega_r^2-\omega_e^2}M_r^H(kl,x,t) + \sum_{s=-1,0}\frac{\omega_e^2}{\omega_s^2-\omega_e^2}M_s^L(kl,x,t); \tag{2.13}$$

M^W is given by the first integral in (2.12), M_r^H is given by

$$M_r^H(kl,x,t) = a\overline{F}_r(kl,t)\int_0^x \frac{[\mu'(x_0)\,\omega_e^2 - \rho g b(x_0)]}{\overline{\mu}_r'(\omega_e^2-\omega_0^2)}\phi_r(x_0)\,(x-x_0)\,dx_0 \quad (r = 1, 2, \ldots, \infty)$$

and M_s^L is given by

$$M_s^L(kl,x,t) = a\frac{\overline{F}_s(kl,t)\,\overline{M}_s(x)}{\omega_s^2\,\mu_s'} \quad (s = -1, 0).$$

For the purposes of discussing the behaviour of the total wave-induced bending moment in the simple beam, the last form, given by (2.13), is particularly useful. We see that M fluctuates with encounter frequency ω_e. If this is close to a natural frequency resonant motions will occur, and it may be expected that large bending moments will be set up. In the low-frequency range these will depend on the behaviour of M_s^{L} $(s = -1, 0)$, whereas in the high-frequency range the term M_r^{H} $(r = 1, 2, \ldots)$ will be of fundamental importance. When such a resonant condition prevails, we shall refer to the phenomenon as 'resonant encounter', following the terminology of Bishop *et al.* (1973). Other cases for which the possibilities of large bending moments arise are associated with those values of kl which lead to large values of M^{W}, M_r^{H} or M_s^{L}. Such a phenomenon is called 'ship–wave matching' in this terminology, since a condition on kl is equivalent to a condition on the ratio, ship length: wavelength. In theory resonant encounter and ship–wave matching may occur together.

It should be noted that the contribution M^{W} is never associated with resonant encounter. It is in fact related to the wave bending moment which is obtained by balancing the beam statically on a wave. We refer to M^{W} as the 'quasistatic' wave bending moment.

In a ship, the phenomena just described are of course modified by damping. This is relatively high in the first two modes, because at these lower frequencies significant energy may be dissipated by surface wave radiation away from the oscillating ship. In the higher modes, however, hydrodynamic damping is generally very low, and since structural damping in the hull girder is also relatively low, large magnification factors apply to the higher frequency resonances. Hence at higher frequencies of encounter the behaviour of the bending moment is predominantly determined by the terms in M_r^{H}. At lower frequencies, however, these terms are negligible.

It is worth while therefore to consider independently the low- and high-frequency characteristics of $M(x, t)$. The bending moment in a uniform beam may most conveniently be examined since the characteristic modes and frequencies in this case are very simply obtainable. Unfortunately, however, certain degeneracies occur for the lowest two modes of a uniform beam, and we must first consider the low-frequency behaviour for a non-uniform beam.

2.4. *Low-frequency wave bending moments*

We have in mind here frequencies considerably below that conventionally considered as the lowest natural vibration frequency of a ship's hull, which corresponds to vertical flexure in the two node mode. In other words we are dealing with the range for which $\omega_{\mathrm{e}} \ll \omega_r$, $r = 1, 2, \ldots$. This means that in the general expression for bending moments, (2.13), we may neglect the contributions from all the higher modes $r = 1, 2, \ldots$, since they will generally be about two orders of magnitude smaller than the remaining terms.

Consider, therefore, the contributions to the bending moment given by the quasistatic term and the terms leading to resonant encounter in the lowest two modes. Thus

$$M(x,t) = M^{\mathrm{W}}(kl,x,t) + \sum_{s=-1,0} \frac{\omega_{\mathrm{e}}^2}{\omega_s^2}\frac{M_s^{\mathrm{L}}(kl,x,t)}{(1-\omega_{\mathrm{e}}^2/\omega_s^2)}. \tag{2.14}$$

This is related in a complex manner to the characteristic functions $\phi_{-1}(x)$ and $\phi_0(x)$. Hence the bending moment is likely to be influenced by the distortions involved in the lowest modes of a non-uniform flexible beam. It seems, however, that for a relatively stiff beam this influence must be small, and it is useful to consider the behaviour of a ship assuming it to act as a rigid beam at these low frequencies. It would appear worth while to investigate subsequently the significance of hull flexibility in the lowest modes.

If the quantity $M^{\mathrm{W}}(kl, x, t)$ is calculated using the characteristic modes of a rigid ship (pitching and heaving) its amplitude is found to be identical to the expression obtained from a quasistatic analysis. In this the ship is in equilibrium in a sinusoidal wave, and the bending moment at a section x is calculated by finding the moment arising solely from the differential buoyancy forces due to the wave. M^{W} is thus related to the orthodox quasistatic bending moment calculation which, modified to allow for the orbital motion of wave particles (the Smith correction), formed the basis of ship stressing for many years. It is for this reason that we have called M^{W} the quasistatic term, even for a non-rigid ship.

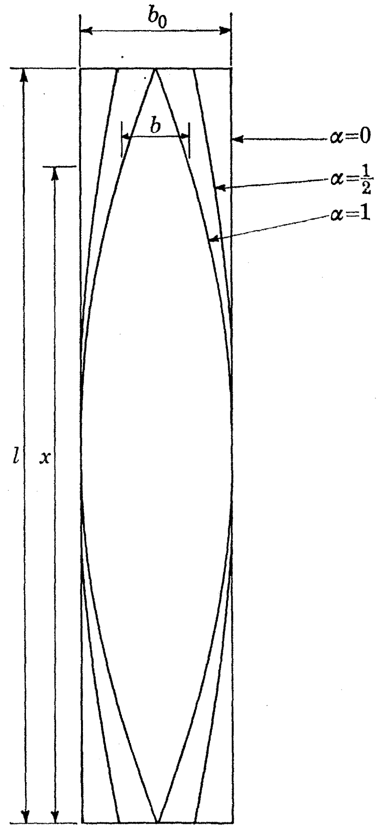

FIGURE 1. Uniform and parabolic planform beams. $b(x) = b_0[1-4\alpha(x/l-\frac{1}{2})^2]$.

Still retaining the assumption of a rigid ship, we see that our simple dynamic analysis contains terms additional to M^{W}. These involve the magnification factors $1/(1-\omega_e^2/\omega_s^2)$, $s = 1, 2$. For a real ship damping plays a very important role in these lower modes introducing, among other effects, a severe limitation on the magnification factor. But before examining in a rudimentary manner this influence of damping, let us consider the undamped bending moment at the midship section of a beam completely symmetric fore and aft. For simplicity in obtaining the analytic expressions we assume a beam of parabolic waterline planform (see figure 1), of rectangular cross-section, in which the virtual mass is taken as being uniformly distributed. Although severe, these restrictions are not such as to remove all semblance of reality from the resulting mathematical model.

We therefore obtain $M(x, t)$ for the case when the quantities μ' and $\rho g b$ are given by

$$\mu'(x) = \text{constant},$$
$$\rho g b(x) = \rho g b_0[1-\alpha(1-2x/l)^2],$$

where b_0 is the maximum breadth of the beam. The analysis is outlined in appendix B, leading to the following definitions of the terms in (2.14):

$$\frac{M^{W}(kl, \frac{1}{2}l, t)}{\rho g b_0 l^2 a} = f_0^{W}(kl) \sin(\omega_e t - \tfrac{1}{2}kl),$$

$$M^{\mathrm{L}}_{-1}(kl, \tfrac{1}{2}l, t) = 0,$$

$$\frac{M^{\mathrm{L}}_0(kl, \frac{1}{2}l, t)}{\rho g b_0 l^2 a} = f^{\mathrm{L}}_0(kl) \sin(\omega_e t - \tfrac{1}{2}kl),$$

where f_0^{W} and f_0^{L} are given in appendix B. The amplitude of the amidship bending moment is then given by

$$\frac{M(\frac{1}{2}l)}{\rho g b_0 l^2 a} = f_0^{\mathrm{W}}(kl) + \frac{\omega_e^2/\omega_0^2}{1-\omega_e^2/\omega_0^2} f_0^{\mathrm{L}}(kl). \tag{2.15}$$

The functions $f_0^{\mathrm{W}}(kl)$ and $f_0^{\mathrm{L}}(kl)$ are plotted in figures 2 and 3 for the cases $\alpha = 1$ and $\alpha = 0.5$ respectively, corresponding to waterline plans indicated in figure 1. In these graphs the abscissa is the ratio of ship length l to wavelength Λ, and in deep water

$$kl = 2\pi \frac{l}{\Lambda} = \omega^2 \frac{l}{g}.$$

The qualitative manner in which the quasistatic term and the resonant encounter term contribute to the total bending moment may be deduced from these curves. But first it is necessary to examine the relative magnitudes of the relevant frequencies, and next to give some consideration to damping in real ships.

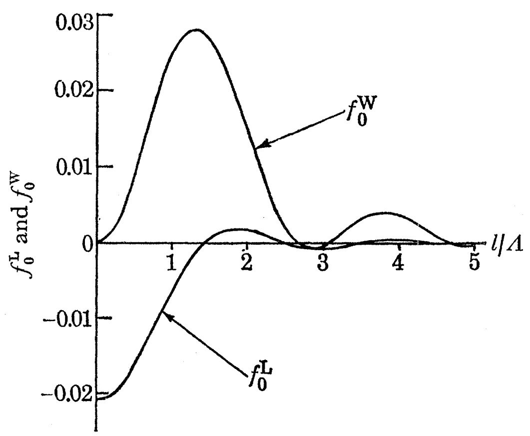

FIGURE 2. Bending moment functions f_0^{W}, f_0^{L} for beam of parabolic planform ($\alpha = 1.0$).

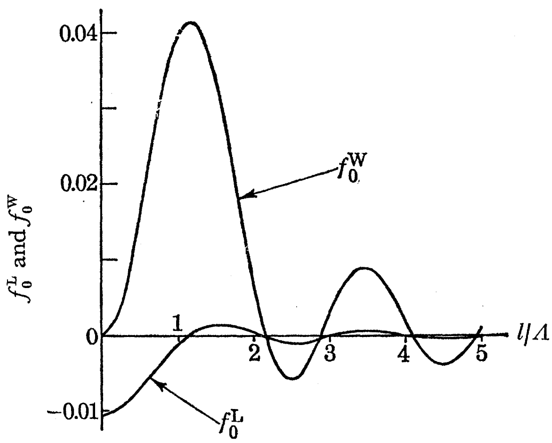

FIGURE 3. Bending moment functions f_0^{W}, f_0^{L} for beam of parabolic planform ($\alpha = 0.5$).

Taking the case $\alpha = 1$ as an example, we see that the maximum value of the quasistatic wave bending moment occurs for a ship length/wavelength ratio of about 1.32. This value corresponds to the condition we have defined above as ship–wave matching, and it is equivalent to a wave frequency given by

$$\omega_{\mathrm{m}} = 2.88\sqrt{(g/l)}.$$

Now the resonant frequency in heave ω_0 depends on the proportions of the ship and on the added mass (itself dependent on frequency for a real ship, though we shall here ignore this added complexity). It appears that, for most conventional ships, ω_0 is of the same order of magnitude as the value ω_{m} giving ship–wave matching, in many cases ω_0 being slightly larger.

Turning to the encounter frequency ω_e, and limiting our considerations to head seas, we observe that if the ship has forward velocity the frequency of encounter corresponding to ship–wave matching ω_{em} is larger than the absolute wave frequency ω_m. Therefore for the cases referred to above, the ship need proceed at only a relatively low speed to reach the condition $\omega_{em} = \omega_0$. Stated another way, the condition for ship–wave matching frequency ω_m to equal wave frequency corresponding to a resonant frequency of encounter ω_0, occurs at relatively low speeds. Let us consider the behaviour near this condition for the particular waterlines given by $\alpha = 1$ and $\alpha = 0.5$.

Referring to figures 2 and 3 we see that in both cases, in the region of the wave frequency ω_m, the function f_0^L is relatively small. For the ideal undamped beam developed here, the smallness of f_0^L would be irrelevant at the resonant condition we are examining, since the magnification would be infinite, and the contribution of the other term f_0^W negligible. But for a real situation this is not the case. In fact the damping due to wave-making at these low frequencies is so high that magnification factors greater than 2.5 are uncommon. Hence the relative magnitudes of f_0^L and f_0^W are highly significant. It appears that for the cases shown in figures 2 and 3, with this high intensity of damping, the resonant contribution in this range is quite small. If this result is general, it seems that, within the limits of our assumptions, at the condition in which ship–wave matching and resonant encounter occur simultaneously, the resonant encounter contribution is not large. It will also be small in the immediate vicinity of this resonant condition. If ω_0 is slightly larger than ω_m, this observation will also apply down to zero ship speeds: the resonant encounter contribution will still be small.

At higher speeds, the resonant encounter frequency would correspond to a wave frequency considerably lower than ω_m. In this range f_0^L is seen to be of the same order as f_0^W. This suggests that the resonant encounter contribution would now become of comparable magnitude to the quasistatic contribution, gaining in significance with further increase in ship speed.

It is instructive to consider an illustration of this behaviour. Without suggesting that the following in any way represents the actual quantitative role of damping due to surface wave generation, we may nevertheless obtain therefrom an indication of the significance of damping which is associated with magnification factors of about 2. We neglect entirely the non-dissipative damping introduced as coupling between the two lowest modes, when the ship is under way. Instead of the expression given in (2.15) for the amplitude of amidship bending moment, consider the quantity

$$\frac{M(\frac{1}{2}l)}{\rho g b_0 l^2 a} = \left| f_0^W(kl) + \frac{\omega_e^2/\omega_0^2}{1-\omega_e^2/\omega_0^2 + 2\mathrm{i}\eta\omega_e/\omega_0} f_0^L(kl) \right|.$$

By adding the imaginary term to the denominator, we have introduced damping in a manner analogous to the introduction of viscous damping into a one degree of freedom spring-mass system. The damping factor is given by η. The modulus of the above complex quantity may be evaluated to give

$$\frac{M(\frac{1}{2}l)}{\rho g b_0 l^2 a} = \left[\frac{\{(1-\omega_e^2/\omega_0^2) f_0^W + (\omega_e^2/\omega_0^2) f_0^L\}^2 + 4\eta^2(\omega_e^2/\omega_0^2)\,(f_0^W)^2}{(1-\omega_e^2/\omega_0^2)^2 + 4\eta^2\omega_e^2/\omega_0^2}\right]^{\frac{1}{2}}. \tag{2.16}$$

For the purposes of illustrating the influence of damping, this quantity is plotted in figure 4 for the waterline given by $\alpha = 1$, with damping factor $\eta = 0.2$. The latter corresponds to a magnification factor of 2.5. The abscissa of figure 4 is the ratio l/Λ, and the curves are for a ship whose natural frequency in heave ω_0 is about 12 % higher than the ship–wave matching frequency.

Each curve corresponds, as indicated, to a different ship speed, expressed in terms of ship Froude number (Fr). The dependence arises since the ratio ω_e/ω_0 is related to ship speed. When the wave frequency corresponding to resonant encounter equals $1.04\omega_m$, the resonant encounter contribution to the amidship bending moment is zero, since in that case $(l/\Lambda) = 1.43$ and $f_0^{\mathrm{L}} = 0$ (see figure 2). For the ship described by figure 4 this corresponds to a Froude number of about 0.025. In the region of this condition, that is to say for Froude numbers up to about 0.05, the influence of ship speed on bending moment remains small. But for higher Froude numbers, $Fr > 0.1$ in this case, the phenomenon of resonant encounter becomes important, and the maximum bending moment increases rapidly as ship speed increases.

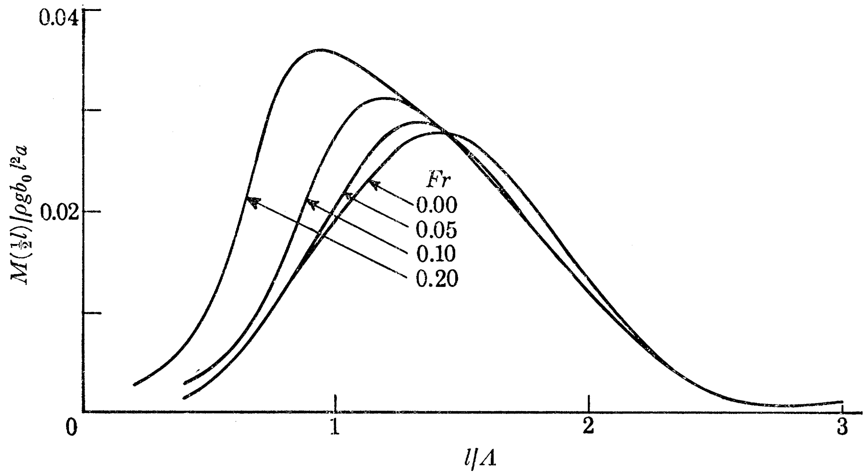

FIGURE 4. Non-dimensionalized low-frequency bending moment for beam of parabolic planform ($\alpha = 1.0$) with damping factor $\eta = 0.2$, at various Froude numbers.

The behaviour of the damped beam illustrated by figure 4 is characterized by the influence of Froude number on the relative significance of ship–wave matching and resonant encounter in the low-frequency range. However, the analysis leading to these predictions is based on an idealization that may seem sufficiently far removed from reality to render useless any general conclusions that might be drawn. Fortunately this does not appear to be the case. On the one hand there are sound theoretical reasons why the quasistatic term is a fair approximation to the total low-frequency bending moment, at low Froude numbers: a consistent perturbation expansion shows that in the equation of motion of a slender body on a free surface, the inertia and damping terms are of second order – hence the predominant contribution to the bending moment arises from the hydrostatic term (Ogilvie 1969; Newman 1970). And, on the other hand, the type of behaviour we have suggested is qualitatively in accordance with observed results, both from full-scale ships and from model tests. The curves of figure 4 are not unlike those that have been measured for a wide range of ship types, of which the literature is amply filled.

We find therefore that at least in the low-frequency range there is some correlation between our very simple mathematical model and the behaviour of real ships. It is hoped that the proposed modal analysis will give sound theoretical explanations of physical effects that are observable throughout the frequency range encountered in practice. There are of course several aspects upon which we may hardly hope to touch in the first stage of development of this analysis, some of which have been totally ignored in the past. One of these is the influence of distortions in the lowest modes: while it is negligible for conventional ships, it may not be so for very long flexible vessels. But fundamental to an analysis of the type suggested here must necessarily be consideration of wave-excited response at higher frequencies. This is discussed in what follows, where the simple theory is further developed for the case of a uniform beam.

2.5. *The special case of a uniform beam*

In the pursuit of a theoretical explanation of the behaviour of ships in waves, there are several advantages to studying the phenomenon for a uniform beam. Not negligible is the fact that the characteristic modes and frequencies of a uniform free–free beam are well known (see, for example, Bishop & Johnson 1960), and the manner in which they are modified by buoyancy forces may be easily calculated. But there is a further point: particularising from non-uniform to uniform beam introduces a degeneracy into the bending moment expression, which simplifies the low-frequency behaviour discussed previously. We have discussed the relative smallness of the resonant contribution to the low-frequency bending moment, at least for certain ships at zero or low speed. But for the uniform beam this contribution is identically zero. Hence consideration of this special case will give a clear illustration of the ship–wave matching phenomenon in the low-frequency range.

It may be seen from (2.4) that the characteristic functions of the beam satisfy

$$\mathrm{d}^4\phi_r/\mathrm{d}x^4 - \beta_r^4\phi_r = 0, \tag{2.17}$$

where
$$\beta_r^4 = \frac{\mu'\omega_r^2 - \rho g b}{EI}.$$

The free–free boundary conditions lead to

$$\frac{\mathrm{d}^3\phi_r}{\mathrm{d}x^3} = 0 = \frac{\mathrm{d}^2\phi_r}{\mathrm{d}x^2} \quad \text{at} \quad x = 0, l. \tag{2.18}$$

Apart from the modified definition of β_r, (2.17) and (2.18) are identical to the equations and boundary conditions for the modes of a free–free beam *in vacuo*. The characteristic functions are:

$$\left.\begin{aligned} r = -1:&\quad \phi_{-1}(x) = (2\sqrt{3}/l)\,(x - \tfrac{1}{2}l) \quad \text{with} \quad \beta_{-1} = 0\\ r = 0:&\quad \phi_0(x) = 1 \quad \text{with} \quad \beta_0 = 0\\ r = 1, 2, \ldots:&\quad \phi_r(x) = \cosh\beta_r x + \cos\beta_r x - \lambda_r(\sinh\beta_r x + \sin\beta_r x)\end{aligned}\right\} \tag{2.19}$$

where
$$\lambda_r = \frac{\cosh\beta_r l - \cos\beta_r l}{\sinh\beta_r l - \sin\beta_r l}$$

and β_r is given by
$$\cos\beta_r l \cosh\beta_r l = 1.$$

The natural frequencies are given by

$$\omega_{-1}^2 = \omega_0^2 = \rho g b/\mu',$$
$$\omega_r^2 = \omega_0^2 + EI\beta_r^4/\mu' \quad (r = 1, 2, \ldots).$$

Whereas the frequencies depend on the added mass and buoyancy, the characteristic functions are identical to those for a uniform beam *in vacuo*. In particular, the lowest two ($r = -1, 0$) correspond to rigid body motions of the beam. Hence the simple uniform beam has two special properties not generally possessed by the non-uniform floating beam: the lowest two modes are free of distortion, and the corresponding natural frequencies are equal. Therefore these modes

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are associated with the indices $r = -1, 0$, in order to distinguish them from all other modes. (The association of $r = 0$ with the mode symmetric about the beam midpoint, and $r = -1$ with the antisymmetric mode, instead of vice-versa, is a fairly arbitrary choice. The present definitions agree with those in Bishop *et al.* 1973.)

The resonant contribution in each of the lowest two modes is related to the quantity M_s^{L} in (2.13), given by

$$M_s^{\mathrm{L}}(kl, x, t) = a\,\frac{\bar{F}_s(kl, t)\,\bar{M}_s(x)}{\omega_s^2\,\mu_s'} \quad (s = -1, 0),$$

where

$$\bar{M}_s(x) = \int_0^x (\mu'\omega_s^2 - \rho g b)\,\phi_s(x_0)\,(x - x_0)\,\mathrm{d}x_0 = EI\phi_s''(x).$$

Since the lowest two modes of a uniform beam are distortion free, the curvatures $\phi_s''(x)$ are zero at all points. Hence $\bar{M}_s$ and M_s^{L} are zero.

This important conclusion may be deduced in another way. $\bar{M}_s(x)$ is seen to be the bending moment at the point x when the beam is oscillated at frequency ω_s in the sth mode on a flat sea. Thus $\bar{M}_s(x)$ is the bending moment due to heaving ($s = 0$) or pitching ($s = -1$) at the appropriate natural frequency in still water. It arises when the buoyancy force is not exactly opposed by the inertia (D'Alembert) force at a section. The integrals of these quantities along the length of the beam must of course balance each other, but for a non-uniform beam any lack of balance at a section gives rise to a bending moment. Now for the uniform beam these quantities are in fact balanced at every point: each section is identical, and the buoyancy exactly opposes the inertia. Hence $\bar{M}_s$ is zero ($s = -1, 0$) for the uniform beam, and so therefore is the low-frequency resonant contribution M_s^{L}.

The total bending moment response of a uniform beam is thus given by

$$M(x, t) = M^{\mathrm{W}}(kl, x, t) + \sum_{r=1}^{\infty} \frac{\omega_{\mathrm{e}}^2 - \omega_0^2}{\omega_r^2 - \omega_{\mathrm{e}}^2} M_r^{\mathrm{H}}(kl, x, t).$$

For a rigid beam (and for a flexible beam at $\omega_{\mathrm{e}} = \omega_0$) the summation is identically zero. The amplitude of wave-induced bending moment in a rigid uniform beam is exactly given by the quasistatic term; in this case there is no dynamic effect at any frequency. But when account is taken of the beam flexibility, resonant encounter becomes possible at frequencies near or above ω_1, the frequency of vibration in the two node mode.

It is important to note why resonant encounter is important for the flexible uniform beam. The amplitude of M_r^{H} ($r = 1, 2, \ldots$) is non-zero, except for certain values of kl as indicated in what follows. Whereas in the lowest modes the hydrodynamic damping is sufficiently large to give a small magnification factor at resonance, this is not the case in the higher modes. At the higher frequencies, damping due to wave generation is small, if not negligible. The primary source of energy dissipation is structural damping, and the magnification in the modes $r = 1, 2, \ldots$ is relatively large.

For the flexible uniform beam the following conditions for large dynamic stresses may be distinguished:

(i) Ship–wave matching, where the ship length/wave length ratio is such as to cause large M^{W}, but the encounter frequency is well below ω_1.

(ii) Resonant encounter, for which the wave encounter frequency is close to a natural frequency ω_r ($r = 1, 2, \ldots$).

But the resonant encounter phenomenon is itself influenced by a degree of ship–wave matching, determined by the variation of M_r^{H} with kl.

These two conditions will be examined separately, the assumption being that at the frequency at which one occurs the contribution of the other is negligible. The simultaneous occurrence of these as independent phenomena in confused seas will be discussed subsequently. First we must show how the operators M^{W} and M_r^{H} vary with wavelength, and we will illustrate this by considering the bending moment amidships.

For the uniform beam the amplitude of M^{W} is given by

$$\frac{|M^{\mathrm{W}}(kl, \tfrac{1}{2}l, t)|}{\rho g b l^2 a} = f_0^{\mathrm{W}}(kl) = \frac{1-\cos\sigma}{4\sigma^2} - \frac{\sin\sigma}{4\sigma},$$

where $\sigma = \frac{1}{2}kl$. This is plotted in figure 5, where it is seen that the absolute maximum occurs for a ship length/wavelength ratio $l/\Lambda = 1.11$. This corresponds to a wave frequency $\omega = 2.65\sqrt{(g/l)}$. Below this wave frequency M^{W} decreases steadily to zero. Above this value, there is a series of smaller relative maxima of $|M^{\mathrm{W}}|$, the first corresponding to a frequency $\omega = 3.88\sqrt{(g/l)}$ and a wavelength given by $l/\Lambda = 2.40$. This graph for the uniform beam is analogous to the graphs for f_0^{W}, in figures 2 and 3, for certain non-uniform beams. But in this case f_0^{L} is of course identically zero for all values of kl.

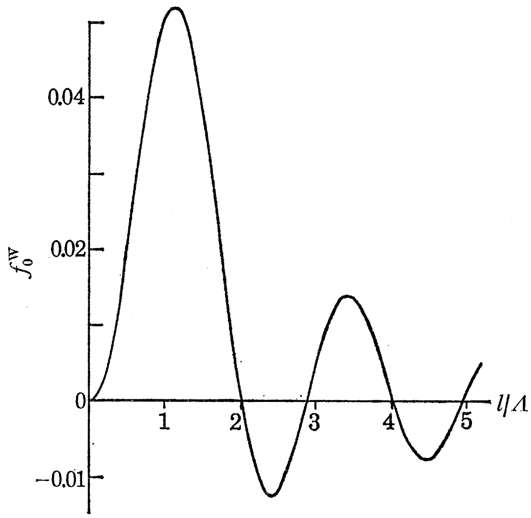

FIGURE 5. Non-dimensionalized low-frequency bending moment function f_0^{W} for uniform beam at all speeds.

To illustrate the higher mode operators M_r^{H} $(r = 1, 2, \ldots)$ we shall here indicate the operator M_1^{H} appropriate to the two node mode. This is of the greatest significance for the resonant encounter phenomenon since damping is generally smallest in this mode. The amidship value is given by

$$M_1^{\mathrm{H}}(kl, \tfrac{1}{2}l, t) = a\bar{F}_1(kl, t)\frac{1}{l}\int_0^{\frac{1}{2}l} \phi_1(x_0)\,(\tfrac{1}{2}l - x_0)\,\mathrm{d}x_0.$$

Using the definition of $\bar{F}_1$ and the known mode shape of the uniform beam, we can easily find the amplitude of this operator. The result is

$$\frac{|M_1^{\mathrm{H}}(kl, \tfrac{1}{2}l, t)|}{\rho g b l^2 a} = f_1^{\mathrm{H}}(kl),$$

where the function f_1^{H} is derived in appendix C. This is plotted in figure 6. The wave frequencies corresponding to resonant encounter in the two node mode are associated with wavelengths typically very much smaller than ship length. Although f_1^{H} has been plotted for values only up to $l/\Lambda = 10$, its behaviour at yet shorter wavelengths is clear: the function fluctuates rapidly, passing through zero at approximately integer values of l/Λ, and the peaks diminish successively. We shall find, however, that the significance of the fluctuations is reduced in a real wave system, when account is taken of slight departures from two-dimensionality of approaching waves.

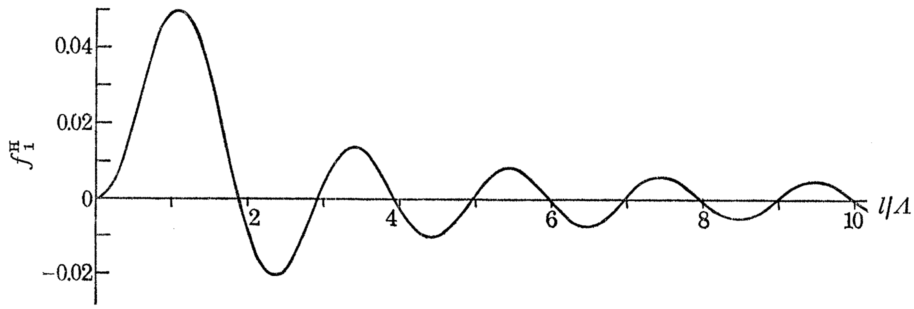

FIGURE 6. Non-dimensionalized high-frequency bending moment function f_1^{H} for uniform beam. $\sigma = \frac{1}{2}kl = \pi l/\Lambda$.

3. Amidship bending moments in a uniform beam in confused seas

3.1. *The sea spectrum*

The operators we have thus far obtained give the amidship bending moment in a uniform beam in progressive sinusoidal waves in deep water. We now use these operators to find mean square bending moments in confused seas, obtaining results both for a long-crested sea and for particular cases of short-crested seas.

The Fourth International Ship Structures Congress (I.S.S.C. 1970) recommended that bending moments be calculated on the basis of a wave spectrum of the form

$$S(\omega) = (A/\omega^5)\,\mathrm{e}^{-B/\omega^4}, \tag{3.1}$$

where A and B are functions both of observed wave height and of observed wave period. However, in order to reduce the number of variables in this basic study, it appeared more convenient to use the International Towing Tank Conference one parameter spectrum (I.T.T.C. 1969). The latter is written in the same form as above, but A and B are functions only of significant wave height $h_{\frac{1}{3}}$ (in metres). In fact

$$\left.\begin{aligned} A &= 8.10\times 10^{-3}g^2,\\ B &= 3.11/h_{\frac{1}{3}}^2, \end{aligned}\right\} \tag{3.2}$$

where g is the gravitational constant in the appropriate units. The form of the resulting wave spectrum is shown in figure 7, where $S(\omega)$ is plotted for a number of values of $h_{\frac{1}{3}}$.

To include the effect of directional spread of wave energy, the I.S.S.C. and I.T.T.C. recommend use of the spectrum

$$S^*(\omega,\theta) = S(\omega)\,C_n\cos^n\theta, \tag{3.3}$$

where θ is the angle between the direction of a wave component and the predominant wave direction. C_n is a normalizing constant. The exponent n is commonly taken as 2 or 4, for which

$$C_2 = 2/\pi, \quad C_4 = 8/3\pi.$$

The higher the value of n, the narrower the directional spread implied. It is suggested in the report (I.S.S.C. 1970) that, to err on the safe side in estimating maximum bending moments, it is advisable to use a high value of n, or to omit the directional spread altogether. Results shown here for the uniform beam, however, suggest that mean square bending moments may in some cases be increased by allowing for spread of wave energy through use of these formulae.

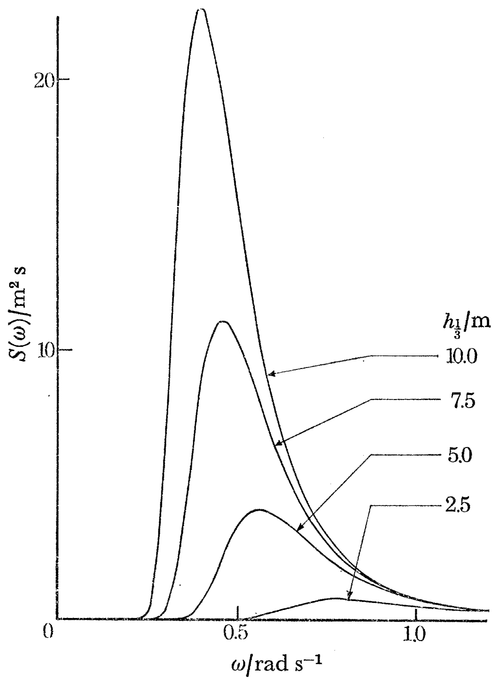

FIGURE 7. Wave spectra for various sea states.

3.2. *Low-frequency wave bending moment response*

The low-frequency amidship bending moment response operator for the uniform beam (or 'uniform ship') in head seas is

$$\frac{|M^{\mathrm{W}}(kl, \tfrac{1}{2}l, t)|}{\rho g b l^2 a} = f_0^{\mathrm{W}}(kl) = \frac{1-\cos\sigma}{4\sigma^2} - \frac{\sin\sigma}{4\sigma},$$

where
$$\sigma = \tfrac{1}{2}kl = \tfrac{1}{2}\omega^2 l/g.$$

But for sinusoidal waves having wavenumber k approaching at an angle θ to the axis of the ship (which is the x-axis), the wave depression along the ship is given by

$$\zeta(x, t) = a\sin(\omega_e t - kx\cos\theta).$$

Therefore for waves from direction θ, the above operator should be used with

$$\sigma = \tfrac{1}{2}kl\cos\theta = (\tfrac{1}{2}\omega^2 l/g)\cos\theta.$$

The bending moment operator f_0^W is then a function of both wave frequency ω and wave direction θ. It does not, however, depend on the encounter frequency, and the low-frequency bending moments in the uniform ship are independent of the speed of the ship.

If this operator is combined with the wave spectrum in the usual manner, we may obtain a mean square bending moment coefficient, the prerequisite to determining long-term expected maximum stresses. It is defined by

$$M_W^{*2} = \frac{1}{l^2}\int_0^\infty \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (f_0^W)^2 S(\omega)\, C_n \cos^n\theta \,d\theta\, d\omega$$
$$= \frac{1}{l^2}\int_0^\infty S(\omega) \left[C_n \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (f_0^W)^2 \cos^n\theta\, d\theta\right] d\omega.$$

Note that M_W^* is dimensionless, since $S(\omega)$ has dimensions of (length$^2\times$time) and f_0^W is non-dimensional. The physical bending moment is given by $\rho g b l^3 M_W^*$. We may write

$$M_W^{*2} = \frac{1}{l^2}\int_0^\infty S(\omega)\,[h_0^W(\omega, n)]^2\, d\omega. \tag{3.4}$$

The quantity h_0^W incorporates the directionality of the wave spectrum. As $n \to \infty$, the wave spectrum tends to the unidirectional case, and $h_0^W \to |f_0^W|$.

The influence of directional spread may be examined by plotting h_0^W against ship length/wave-length ratio (or wave frequency) for different values of the exponent n. This is shown in figure 8, which includes the cases $n = 2, 4, \infty$. The integral was evaluated by Simpson's rule, using step sizes of 1°. The form of the primary peak near $l/\Lambda = 1$ is seen to be reasonably constant, whether or not directional spread is allowed for. However, the secondary peaks which we have found for a long-crested sea ($n = \infty$) are virtually eliminated in a widely spread sea ($n = 2$).

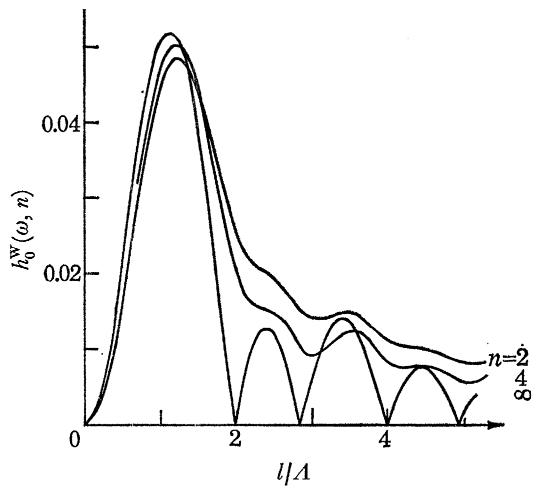

FIGURE 8. Low-frequency bending moment function h_0^W for various wave energy directional spread exponents n.

We may understand the reason for this latter behaviour by considering what happens at $l/\Lambda = 2$ ($\sigma = 2\pi$), which in a unidirectional head sea gives $f_0^W = 0$ (figure 5). Thus $f_0^W = 0$ for $\omega = \sqrt{(4\pi g/l)}$. In a directional sea this same value of ω corresponds to values of σ ranging from 2π to 0, as θ varies from 0 (ahead) to $\pm\frac{1}{2}\pi$ (abeam). Between these limits on σ, f_0^W is positive, and consequently $h_0^W(\omega, n)$ has a non-zero value for all non-zero ω if n is finite. The peak to trough fluctuations are reduced as n is decreased, since the behaviour away from $\theta = 0$ then increases in importance.

The dependence of h_0^W upon the law chosen to represent directional spread in turn influences the value calculated for the quantity M_W^*. This has been computed for a uniform ship of length 308 m in waves of significant height 5 m, an example used subsequently in the investigation of high-frequency response. The integration over the frequency range ω was performed by Simpson's rule, truncated at $\omega = 1.285$ rad/s. Above this value of ω, the integrand is less than 0.025 % of its maximum value, and its contribution to M_W^* is negligible. The results indicate that the assumption of a unidirectional sea leads to an *under*estimate of about 10 % for the amidship bending moment in the uniform ship, as compared with the value obtained using a $\cos^2\theta$ wave energy distribution.

The magnitude of M_W^* is strongly dependent on sea state, as measured by significant wave height. This is due to two effects. An increase in $h_{\frac{1}{3}}$ causes an increase in $S(\omega)$ over the whole range of ω (as shown in figure 7), which in turn must increase M_W^*. But a change in $h_{\frac{1}{3}}$ also shifts the frequency about which maximum wave energy is centred: this frequency decreases as significant wave height increases. Hence the magnitude of $h_{\frac{1}{3}}$ determines how near the peak in $S(\omega)$ is to the peak in $h_0^W(\omega, n)$, hence also influencing M_W^*. In extremely heavy seas, the maximum energy is concentrated in the region of such large wavelengths that the value of $h_0^W(\omega, n)$ is relatively small.

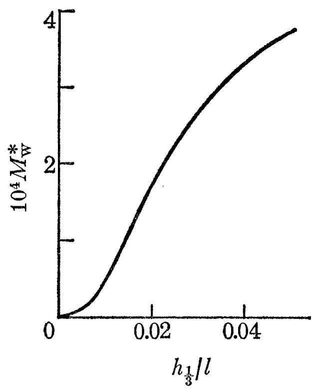

FIGURE 9. Variation of dimensionless r.m.s. low-frequency bending moment with sea state.

Thus there is a gradual levelling off of M_W^* at high values of $h_{\frac{1}{3}}$. This behaviour is illustrated in figure 9, which is a plot of M_W^* against the dimensionless wave height $(h_{\frac{1}{3}}/l)$, computed using $n = 2$. The figure gives a single curve valid for any geometry of the uniform ship, as we would expect from consideration of the definition of M_W^* in (3.4), and the dependence of $S(\omega)$ on $h_{\frac{1}{3}}$ given by (3.1) and (3.2). The quantity h_0^W in (3.4) is a function of σ, hence of $\omega^2 l$. The equation for M_W^* may therefore be written

$$M_W^{*2} = \int_0^\infty \frac{A}{(\omega\sqrt{l})^5}\left[\exp\left(\frac{-3.11}{\omega^4 l^2}\right)\left(\frac{l}{h_{\frac{1}{3}}}\right)^2\right][h_0^W(\omega\sqrt{l}, n)]^2\sqrt{l}\,d\omega.$$

Transforming the integration variable from $\omega\sqrt{l}$ to ω', we see that the functional relationship between M_W^* and $(h_{\frac{1}{3}}/l)$ is independent of the length of the uniform ship.

The graph in figure 9 gives the dependence of low-frequency bending moment on sea state, for a uniform ship of any length or breadth. However, the compact representation of the information in this figure to some extent conceals its practical significance. If M_W^* were plotted against $h_{\frac{1}{3}}$, rather than the dimensionless wave height, the importance of ship size would be better illustrated. We should find that, for a given sea state, M_W^* decreases as ship length l increases. The resulting graphs would be analogous to those of Little & Lewis (1971), in which were plotted the variation of r.m.s. wave bending moment coefficients (related to M_W^*) with Beaufort number. These were deduced from extensive measurements on five ships in service conditions, but they show very similar characteristics to those we have derived from the simple theoretical approach.

3.3. *High-frequency wave-excited vibratory bending moment response*

The high-frequency amidship bending moment for the uniform beam in head seas is

$$\sum_{r=1}^{\infty} \frac{\omega_e^2-\omega_0^2}{\omega_r^2-\omega_e^2} M_r^{\mathrm{H}}(kl, \tfrac{1}{2}l, t).$$

For a real ship only the lowest mode ($r = 1$) is likely to be of great significance, so it is reasonable to illustrate the high-frequency behaviour by considering the amplitude of the first term in the above series. Thus we consider

$$\frac{\omega_e^2-\omega_0^2}{\omega_1^2-\omega_e^2}|M_1^{\mathrm{H}}(kl, \tfrac{1}{2}l, t)| = \rho g b l^2 a \frac{\omega_e^2-\omega_0^2}{\omega_1^2-\omega_e^2} f_1^{\mathrm{H}}(\sigma)$$

where f_1^{H} is the quantity plotted in figure 6.

This expression has been derived neglecting all damping. But in order to derive mean square bending moments analogous to those obtained for the low-frequency contribution (which is considered as being independent of the term now under discussion), we must make some assumption about damping. It seems reasonable to assume that at these higher frequencies dissipative damping due to hydrodynamic effects is negligible. Only structural damping is significant, and in the calculations which follow this is assumed to limit the reasonant magnification factor in the two node mode to 100. This is of the order of measurements made on full-scale welded hulls (Aertssen & de Lembre 1971), and corresponds to a damping factor $\eta = 0.005$.

The above expression for bending moment operator is therefore modified to allow for damping, by using

$$H_1(\omega_e) = \frac{\omega_e^2-\omega_0^2}{\omega_1^2[(1-(\omega_e^2/\omega_1^2))^2+4\eta^2(\omega_e^2/\omega_1^2)]^{\frac{1}{2}}} \tag{3.5}$$

in place of the factor $(\omega_e^2-\omega_0^2)/(\omega_1^2-\omega_e^2)$. As before, for waves from a direction θ

$$\sigma = (\tfrac{1}{2}\omega^2 l/g)\cos\theta.$$

Thus $f_1^{\mathrm{H}}(\sigma)$ is a function of ω and θ.

The corresponding mean square bending moment coefficient is

$$M_{\mathrm{H}}^{*2} = \frac{1}{l^2}\int_0^{\infty}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} [H_1(\omega_e)]^2[f_1^{\mathrm{H}}(\sigma)]^2 S^*(\omega,\theta)\,\mathrm{d}\theta\,\mathrm{d}\omega.$$

This may be rewritten by relating the encounter frequency ω_e and the wave frequency ω. We have

$$\omega_e = \omega[1+(U\omega/g)\cos\theta],$$

where U is the forward speed of the ship. The solution of this last equation, for waves from ahead, is

$$\omega = \frac{[1+4\omega_e(U/g)\cos\theta]^{\frac{1}{2}}-1}{2(U/g)\cos\theta}.$$

Hence $$\sigma = \frac{gl}{8U^2\cos\theta}\{[1+4\omega_e(U/g)\cos\theta]^{\frac{1}{2}}-1\}^2.$$

Mapping wave frequency ω into encounter frequency ω_e in the integral, and using

$$S^*(\omega,\theta)\,\delta\omega\,\delta\theta = S^*(\omega_e,\theta)\,\delta\omega_e\,\delta\theta,$$

we have $$M_H^{*2} = \frac{1}{l^2}\int_0^\infty [H_1(\omega_e)]^2\left\{\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}[f_1^H(\omega_e,\theta)]^2 S^*(\omega_e,\theta)\,d\theta\right\}d\omega_e.$$

Now $$S^*(\omega_e,\theta) = \frac{S^*(\omega,\theta)}{1+2\omega(U/g)\cos\theta} = \frac{S(\omega)\,C_n\cos^n\theta}{[1+4\omega_e(U/g)\cos\theta]^{\frac{1}{2}}}. \tag{3.6}$$

Since, for constant encounter frequency ω_e, $S(\omega)$ may be expressed as a function of θ, the mean square bending moment may be evaluated from

$$M_H^{*2} = \frac{1}{l^2}\int_0^\infty [H_1(\omega_e)]^2\left\{C_n\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}[f_1^H(\omega_e,\theta)]^2\frac{S(\omega)}{[1+4\omega_e(U/g)\cos\theta]^{\frac{1}{2}}}\cos^n\theta\,d\theta\right\}d\omega_e.$$

This may formally be written

$$M_H^{*2} = \frac{1}{l^2}\int_0^\infty [H_1(\omega_e)]^2 S_1(\omega_e,U)\,d\omega_e. \tag{3.7}$$

The form of M_H^* is fundamentally different from M_W^*, reflecting the different attributes of low-frequency ship–wave matching and high-frequency resonant encounter. Not only is the magnification factor $H_1(\omega_e)$ introduced, but the influence of U is also now strongly apparent.

It is useful to consider first the case of zero ship speed in order to analyse the influence of directional spread exponent n. For this case $\omega_e = \omega$, and we have

$$\begin{aligned} M_H^{*2} &= \frac{1}{l^2}\int_0^\infty [H_1(\omega)]^2 S(\omega)\left\{C_n\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}[f_1^H(\omega,\theta)]^2\cos^n\theta\,d\theta\right\}d\omega \\ &= \frac{1}{l^2}\int_0^\infty [H_1(\omega)]^2 S(\omega)\,[h_1(\omega,n)]^2\,d\omega. \end{aligned}$$

As for the low-frequency case, a quantity (h_1) has been defined which tends to $|f_1|$ as $n\to\infty$. It is plotted against ship length/wavelength ratio in figure 10, for the cases $n = 2, 4, 16, \infty$. Since for the high frequencies appropriate to resonant encounter the wavelength is considerably shorter than the length of any ship, we are only concerned with the characteristics of h_1 for values of l/Λ considerably greater than unity. Figure 10 covers the range near $l/\Lambda = 40$, which applies to the particular example considered below.

In figure 10 it is striking how small is the change in h_1 as the exponent n is increased from 4 to 16. In fact for the curve of h_1 to have fluctuations approaching the same order as those of $|f_1|$, n must be very large indeed, according to the computations made while obtaining the results of figure 10. This may probably be explained as follows. The distance between successive maxima in the $|f_1|$ curve corresponds in this range to a change of σ of about 2 %. However, σ is proportional to $\cos\theta$, so that a 1 % change in σ corresponds to approximately a 1° angle between ship heading and wave direction. For there to be large fluctuations in h_1, the multiplier $\cos^n\theta$ must fall from a maximum to almost zero in half the distance between the peaks of $|f_1|$; that is to say, as θ increases from 0 to 1°. This implies very large values of n. Its practical significance is that a wave system would have to possess negligible directional harmonics to lead to an operator even approaching the ideal two-dimensional result based on $|f_1|$. In a real case it seems likely that small directional components will lead to functions resembling h_1, as plotted in figure 10 for $n = 4$ or 16, much more closely than $|f_1|$. This is not to say, however, that the fluctuations are virtually eliminated by these directional components: indeed they remain highly significant, even though relatively small compared with the fluctuations in the operator $|f_1|$ for the unidirectional case.

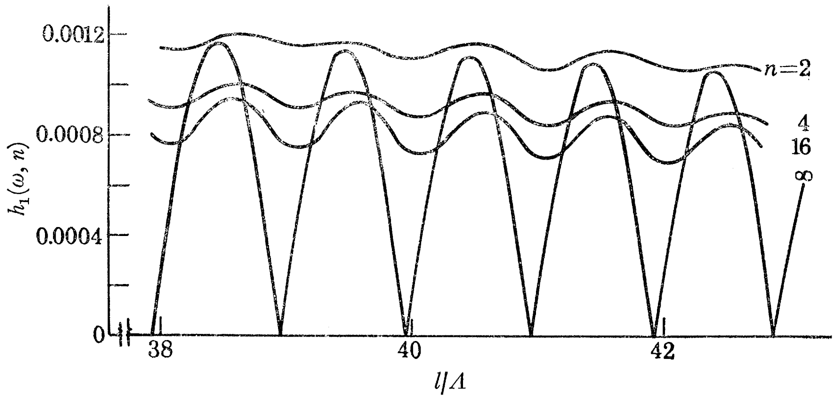

FIGURE 10. High-frequency bending moment function h_1 for various wave energy directional spread exponents n.

When $U \neq 0$, it is not possible to define a quantity analogous to h_1, and the significance of directional spread may not be illustrated in the same manner. It seems reasonable to expect that its influence on M_H^* will not differ qualitatively from what we have described. In subsequent calculations, use is made of the $\cos^2$ law ($n = 2$), for zero and for non-zero ship speeds.

The characteristics of the high-frequency bending moment, and its dependence on speed and sea state, may be investigated by making calculations for the example uniform ship to which we referred previously. It is of length 308 m, having natural frequencies $\omega_0 = 0.57$ rad/s and $\omega_1 = 2.86$ rad/s. These values of l and ω_0 are chosen for convenience since they correspond to the properties of a 200 000 t (d.wt.) tanker analysed by Goodman (1971) – ω_0 was not specified, but the above value is considered a reasonable estimate, and its influence on the results computed here is only secondary. Where possible, the results are generalized by expressing them in non-dimensional form, as in figure 9 for the low-frequency bending moment. However, because of the appearance of ω_1 and ω_0 in the high-frequency expressions, we must introduce two new dimensionless parameters. We choose

$$\sigma_1 = \frac{\omega_1^2 l}{2g}, \quad \tau = \frac{\omega_0}{\omega_1},$$

and M_H^* may then be calculated for specific sets of these parameters. For the uniform ship which approximates to the 200 000 t (d.wt.) tanker, the relevant values are

$$\sigma_1 = 128, \quad \tau = 0.2 \quad \text{(ship T2)}.$$

We have also made some calculations for two other sets of values:

$$\sigma_1 = 289, \quad \tau = 0.2 \quad \text{(ship T1)},$$

$$\sigma_1 = 66, \quad \tau = 0.2 \quad \text{(ship T3)}.$$

Each ship type (e.g. T1) represents a family of uniform ships whose lengths and frequencies generate the appropriate dimensionless parameters. Examples of ships T1 and T3 might have the properties

$$\text{T1: } l = 215\,\text{m}, \quad \omega_1 = 5.14\,\text{rad/s}, \quad \omega_0 = 1.03\,\text{rad/s};$$

$$\text{T3: } l = 400\,\text{m}, \quad \omega_1 = 1.80\,\text{rad/s}, \quad \omega_0 = 0.36\,\text{rad/s}.$$

The natural frequency ω_1 for T3 has deliberately been chosen on the low side, relative to conventional ship sizing, to illustrate the characteristics of a relatively flexible ship.

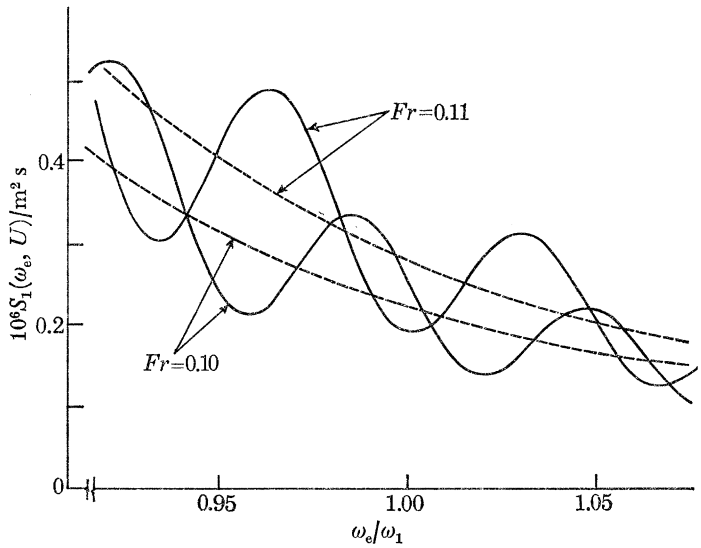

FIGURE 11. Transformed wave spectra $S_1(\omega_e, U)$ for speeds corresponding to Fr = 0.10 and 0.11. ——, Calculated values; – – –, 'averaged' values.

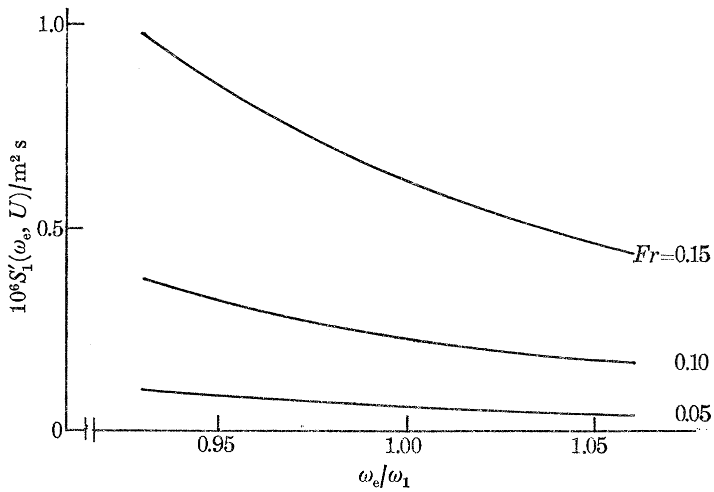

FIGURE 12. Averaged transformed wave spectra at various Froude numbers.

The results of the computations are shown in figures 11 to 15. In figure 11 the function $S_1(\omega_e, U)$, introduced in (3.7), is plotted for ship T2 for two relatively close speeds, which correspond respectively to $Fr = 0.10$ and 0.11. The abscissa has been non-dimensionalized with respect to the natural frequency ω_1. The figure refers to a sea state represented by the non-dimensional wave height $(h_{\frac{1}{3}}/l) = 0.01623$, which for the 308 m ship is equivalent to $h_{\frac{1}{3}} = 5\,\text{m}$.

An examination of figure 11 suggests that the $Fr = 0.11$ curve is approximately equivalent to the $Fr = 0.10$ curve shifted to the right, and increased by a small factor. But because of the fluctuations of $S_1(\omega_e, U)$, this shift causes a marked alteration to its value at the resonant condition $\omega_e = \omega_1$. Hence the amidship bending moment response, directly related to $S_1(\omega_e, U)$, will be appreciably modified by relatively small changes in the forward speed of this theoretical model. For real ships the significance of this phenomenon is an open question. In high waves a ship would probably be unable to maintain constant speed within the accuracy implied by a practical realization of these fluctuations. But in moderate seas, if significant high-frequency stresses occur, it seems possible that a small change in ship speed could cause an appreciable alteration in stress level.

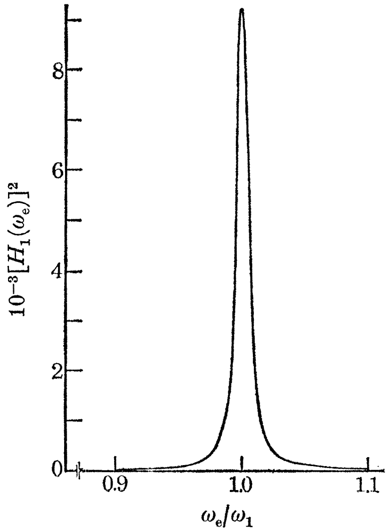

FIGURE 13. Magnification $[H_1(\omega_e)]^2$ for dampling factor $\eta = 0.005$.

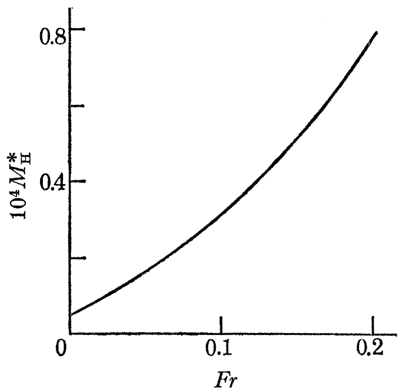

FIGURE 14. Variation of dimensionless r.m.s. high-frequency bending moment with Froude number, for ship type T2. $h_{\frac{1}{3}}/l = 0.01623$.

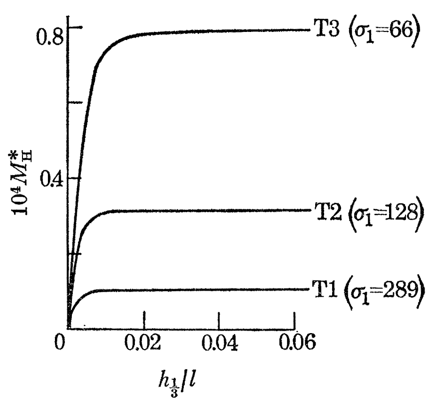

FIGURE 15. Variation of dimensionless r.m.s. high-frequency bending moment with sea state, at $Fr = 0.10$, for various ship types.

Of great importance is the general trend superimposed on this fluctuating behaviour. As well as the shift to the right as Fr increases, there is an overall increase in the magnitude of $S_1(\omega_e, U)$. This is because a given encounter frequency corresponds at higher speeds to lower wave frequencies, hence to higher wave energies. The increase is illustrated by the dashed lines in figure 11, drawn through 'average' values of successive maxima and minima, for the two cases

$Fr = 0.10$ and $Fr = 0.11$. (The 'averages' were computed on the basis of a least squares polynomial fit. The choice of polynomial was somewhat arbitrary, but the results are useful for illustrative purposes.) A family of corresponding average curves is shown in figure 12 for ship T2, for values of Fr from 0.05 to 0.15. At a given encounter frequency, the increase of $S_1(\omega_e, U)$ with Fr is dramatic.

In order to obtain the mean square high-frequency bending moment, given by (3.7), we require the factor $[H_1(\omega_e)]^2$. This is plotted in figure 13 for the example uniform ships (for which $\tau = \omega_0/\omega_1 = 0.2$). The damping factor is assumed to be $\eta = 0.005$. Such relatively light damping leads to a very narrow band response in the vicinity of $\omega_e/\omega_1 = 1$. Comparison of figures 11 and 13 shows that the bandwidth of $H_1(\omega_e)$ is somewhat smaller than the frequency difference between successive maxima and minima of the curves for $S_1(\omega_e, U)$. Hence the process of integrating will not smooth out the fluctuations: the value of M_H^* will be critically dependent on U. A curve of M_H^* against Fr would display significant fluctuations superimposed upon a value steadily increasing with Fr.

It may be that meaningful results are obtained by using the average functions plotted in figure 12 in place of $S_1(\omega_e, U)$ in the integral. Let such a function be $S_1'(\omega_e, U)$. It varies very little within the bandwidth of $H_1(\omega_e)$. It is a reasonable approximation to take

$$M_H^{*2} \simeq \frac{S_1'(\omega_1, U)}{l^2}\int_0^\infty [H_1(\omega_e)]^2\,\mathrm{d}\omega_e$$
$$= \frac{S_1'(\omega_1, U)}{l^2}\left(1-\frac{\omega_0^2}{\omega_1^2}\right)\omega_1\frac{\pi}{4\eta}.$$

(See for example the discussion by Thomson (1965) of lightly damped systems subject to broad band excitation.) In this way we may obtain a good approximation to the mean square response, without having to perform the integration in the frequency domain. Values of M_H^* obtained from this approximation are plotted for different ship Froude numbers in figure 14. This figure represents results for ship T2 in a seaway given by $h_{\frac{1}{3}}/l = 0.01623$. It illustrates most clearly the rapid rise in amidship vibratory bending moment as forward speed of the ship increases.

The dependence of M_H^* upon sea state is of particular interest. This may readily be found by computing $S_1'(\omega_1, U)$ for a given value of Fr and a series of values of $(h_{\frac{1}{3}}/l)$: figure 15 shows the results for ships T1, T2 and T3 at $Fr = 0.1$. We see that, apart from the trivial range of very small $(h_{\frac{1}{3}}/l)$ and insignificant wave excitation, M_H^* is almost independent of sea state, for a given ship. This may also be seen from figure 7, which indicates very little change in wave energy content at the higher frequencies, as $h_{\frac{1}{3}}$ increases. It follows that if wave-excited vibratory moments are at all significant, for a given ship at a given speed, they attain significance in relatively light seas. If valid, this is a most important conclusion. It indicates contrasting characteristics at low and high frequencies. And it suggests that the rapid fluctuations in bending-moment amplitude, superimposed upon the average values which increase with ship's speed, may indeed be relevant to the case of a real ship, if the resulting stresses can reach significant values in moderate seas. Consequently a slight change in forward speed (either an increase or a decrease) may be sufficient to reduce considerably the high-frequency stresses. This has in fact been observed during full-scale measurements of a Great Lakes ore carrier (Hoffman 1972).

To assess the relative significance of wave vibration stresses, compared with low-frequency wave bending stresses, we examine values of M_H^* in moderate seas for the three families of ships T1, T2 and T3. Figure 15 shows that wave vibration stresses increase rapidly as σ_1 decreases; in

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other words, these stresses are more likely to be significant for large and more flexible ships than for small and stiffer ships. Now figure 14 gives the dependence of M_H^* on Fr for ship T2. It was computed for seas given by $h_{\frac{1}{3}}/l = 0.01623$, but figure 15 indicates that the results are approximately valid for all sea states above this. We may deduce from these graphs that in moderate seas the ship T3 need only proceed at a speed equivalent to $Fr = 0.10$ to achieve a value of M_H^* as high as that pertaining to ship T2 at $Fr = 0.20$. This is a striking result when compared with the characteristics of the low-frequency wave bending moment. M_W^* is the same for the two ships in corresponding seas, at all speeds. Furthermore, comparison of figures 15 and 9 demonstrates that the value of M_H^* for ship T3 at $Fr = 0.10$ is approximately equal to M_W^* in seas given by $h_{\frac{1}{3}}/l = 0.0127$; for a 400 m ship this is equivalent to a significant wave height of about 5 m. These results illustrate how wave-excited vibratory stresses, in very large relatively flexible ships, may be of comparable magnitudes to normal wave stresses.

4. Conclusions based on the simple beam analysis

4.1 *The mechanism of wave excitation*

In order to illustrate the development of the simple beam theoretical model, the phenomenon of wave excitation has been examined in some detail. Within the limitations of the theory, discussed in the following section, this leads to a number of observations about the behaviour of a flexible ship in waves.

When placed in a progressive sinusoidal wave approaching from ahead, this ship will respond in a manner determined by the frequency of the oncoming waves. Broadly speaking, there are two independent response mechanisms of significance, corresponding to low and high wave frequencies. Let us first consider what may be described as the 'low'-frequency response. It is excited by waves at frequencies corresponding to wavelengths of the same order as the ship length. The frequency of encounter with such waves is often of the same order as the lowest two resonant frequencies of the ship, corresponding to heave and pitch.† But because of very heavy damping, which results from waves generated by the vertical motions of the ship, magnification of the response at either resonant frequency is small. The behaviour of both motions and stresses is dominated by a phenomenon which has been termed ship–wave matching.

This matching occurs at a certain wave frequency; it corresponds to a wavelength bearing a particular relation to the ship length such that maximum excitation occurs. Hence ship–wave matching is independent of the speed of the ship. The vertical motions of the ship depend on both the degree of ship–wave matching and, to a lesser extent, on the proximity to the condition of resonant encounter (at which encounter frequency coincides with heave or pitch natural frequency). Generally speaking the stresses also depend on these phenomena; but the contribution to the stresses due to resonant encounter is often small at low ship speeds, and is identically zero for the special case of the 'uniform' ship. Hence at low speeds these stresses are not strongly dependent on Froude number.

Consider now the ship's response to waves in the 'high'-frequency range. Such waves are typically much shorter than the ship's length, and their encounter frequencies are of the order of, or higher than, the natural frequency of the lowest symmetric vibration mode (the two-node mode). Because damping at these frequencies is very light, the response is dominated by the

† Note that for a non-uniform flexible ship, neither of these two modes is distortion free.

degree of proximity to resonant encounter in the two node mode (or higher modes). The simple theory indicates a significant dependence on the precise value of ship length/wavelength ratio (ship–wave matching in the 'high'-frequency range), but reasons have been suggested why this would be less important for a real ship in waves. It has been shown that the frequency of the oncoming waves does not therefore greatly influence the magnitude of the resulting 'high'-frequency stresses in a direct manner. Rather it is the encounter frequency that is significant. Thus for a given wave height these stresses are strongly dependent on Froude number.

Although they will not be discussed here further, we must make mention of other high-frequency responses in waves, induced by a somewhat different mechanism. They may be thought of as hybrids of the low- and high-frequency phenomena we have described. For example, when a ship is excited into large pitching motions by low-frequency regular waves, through ship–wave matching, its bow may periodically emerge from the water. Its return later in the same cycle is accompanied by a sudden impulse transmitted from the bow along the hull length, which in turn excites vibrations in the two-node and higher modes. This is the phenomenon of slamming. Somewhat similar is the effect of vibratory forces which may be induced in a pitching or heaving ship, even without its bottom leaving the free surface: the forces are related to the rates of change of fluid momentum which arise when the hull is not wall-sided. It seems likely that the wave-excited vibrations observed in a 47000 t (d.wt.) tanker (Bell & Taylor 1968) were a manifestation of this latter phenomenon, often called whipping. Both slamming and whipping, although excited by a seaway which is assumed to be a stationary random process, are non-stationary processes (Apostolov 1969).

The high-frequency wave-excited response with which we are concerned, the phenomenon which has been called springing, is a stationary phenomenon in a random sea. It has been analysed by Goodman (1971) and van Gunsteren (1970). The results given by these authors refer to the springing phenomenon in a unidirectional seaway. We have seen, however, that such theoretical results are very sensitive to ship's speed. Although the trend is for wave-excited vibratory stresses to increase rapidly with forward speed, the increase is not monotonic: small increases in speed may lead to dramatically lower stresses, if a unidirectional sea is assumed. Incorporation into the theory of minimal directional spread of wave energy reduces the fluctuating nature of the dependence of stress on speed, but does not eliminate this characteristic. A full explanation of the behaviour of real ships must probably await a theory which takes into account the variations in forward speed due to surging in waves.

The influence of directional wave energy spread on low-frequency wave bending moments has also been illustrated by the simple theory developed herein. It appears that by considering only the case of unidirectional seas, the analyst may underestimate the long-term wave-induced stresses.

For a ship in head seas, we may summarize the differences between the low- and high-frequency stress responses, which have been derived from the simple beam mathematical model, as follows. Low-frequency stresses are, at least at low speeds, almost independent of ship Froude number; in general they increase rapidly as sea state deteriorates. High-frequency springing stresses, if at all important, will be evident in moderate seas and will not be significantly higher in much heavier seas. They can be dramatically reduced by making a considerable reduction in ship speed: but it is also possible that in moderate seas they may be reduced considerably by only a slight increase or decrease in speed. In a random seaway the low- and high-frequency phenomena occur simultaneously. Estimates of probable maximum stresses must be based on their combined effects, assuming each to be independent and Gaussian distributed (Miles 1970). We have shown that it is reasonable to expect the relative contribution of low-frequency stresses, for a reference sea state, to decrease, and that of high-frequency stresses to increase, as ships become larger.

4.2. *The need for an improved model*

The model has proved useful in discussing the qualitative behaviour of ships in head seas, with particular reference to factors influencing amidship bending moments and stresses. Clearly this type of analysis in terms of modal responses is also relevant to calculations of hull vibration amplitudes due to propeller excitation or other mechanical exciting forces, at least in the frequency range below that at which the hull ceases to act as a beam with rigid cross-sections.

Our analysis has, for clarity, neglected the effects of shear deformation and rotary inertia, but inclusion of these would not alter the general behaviour which we have described. The existence of distortions in the lowest modes of a ship has been noted, but not considered in detail. All distortions are limited by structural damping, which has been treated in only a rudimentary manner. However, it seems unlikely that a sophisticated consideration of this factor would appreciably modify the observations set out above. Far more serious is the superficial manner in which hydrodynamic damping has been introduced.

The nature of the hydrodynamic forces has been drastically simplified in this paper, in an attempt to obtain a clearer understanding of the structural behaviour of a ship in waves. It is this simplification which imposes the most severe limitations on the application of these ideas to obtain quantitative results for real ships. To improve the theory we must examine the hydrodynamic boundary value problem, attempting to take account of the three-dimensional nature of the flow, and the influence of the free surface. The problem is currently under consideration.

Appendix A. Identities required in derivation of the bending moment $M(x, t)$ in the simple beam

Consider the quantity

$$\bar{M}_r(x) = \int_0^x [\mu'(x_0)\,\omega_r^2 - \rho g b(x_0)]\,\phi_r(x_0)\,(x-x_0)\,\mathrm{d}x_0.$$

Since the characteristic functions $\phi_r(x)$ satisfy

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2\phi_r}{\mathrm{d}x^2}\right) + \rho g b\phi_r - \mu'\omega_r^2\phi_r = 0$$

it may be written

$$\bar{M}_r(x) = \int_0^x \frac{\mathrm{d}^2}{\mathrm{d}x_0^2}\left(EI\frac{\mathrm{d}^2\phi_r}{\mathrm{d}x_0^2}\right)(x-x_0)\,\mathrm{d}x_0.$$

This may be integrated by parts to give

$$\bar{M}_r(x) = \left[\frac{\mathrm{d}}{\mathrm{d}x_0}\left(EI\frac{\mathrm{d}^2\phi_r}{\mathrm{d}x_0^2}\right)(x-x_0)\right]_0^x + \left[EI\frac{\mathrm{d}^2\phi_r}{\mathrm{d}x_0^2}\right]_0^x.$$

Now the boundary conditions at the end $x_0 = 0$ of the non-uniform free-free beam are

$$\frac{\mathrm{d}}{\mathrm{d}x_0}\left(EI\frac{\mathrm{d}^2\phi_r}{\mathrm{d}x_0^2}\right) = 0 = EI\frac{\mathrm{d}^2\phi_r}{\mathrm{d}x_0^2} \quad \text{at} \quad x_0 = 0.$$

Hence we obtain the first identity

$$\bar{M}_r(x) = EI(x)\,\phi''_r(x). \tag{A 1}$$

Next consider the series representation

$$\rho g b(x) \sin(\omega_e t - kx) = \mu'(x) \sum_{r=-1}^{\infty} f_r(kl, t)\,\phi_r(x),$$

where the coefficients $f_r(kl, t)$ are as yet unknown. They may be found by multiplying the equation by $\phi_s(x)$, integrating over the range $0 \leqslant x \leqslant l$, and using the orthogonality property of the functions $\phi_r(x)$. We obtain

$$\int_0^l \rho g b(x)\,\phi_r(x) \sin(\omega_e t - kx)\,\mathrm{d}x = f_r(kl, t)\,\bar{\mu}'_r \quad (r = -1, 0, 1, 2, \ldots).$$

But the left-hand side of the above is $\bar{F}_r(kl, t)$, and we have thus found that

$$\rho g b(x) \sin(\omega_e t - kx) = \mu'(x) \sum_{r=-1}^{\infty} \frac{\bar{F}_r(kl, t)}{\bar{\mu}'_r}\,\phi_r(x). \tag{A 2}$$

This is the second identity we require to obtain the bending moment.

Appendix B. Amidship wave bending moments in a symmetric rigid beam

Advantage may profitably be taken of the condition of symmetry in this special case by transforming the origin to the amidship section, and using the non-dimensional coordinate $\xi = (x/l)$. We consider the case for which

$$\mu'(\xi) = \text{constant},$$

$$\rho g b(\xi) = \rho g b_0(1 - 4\alpha\xi^2).$$

Symmetry arguments show that, of the two rigid body modes, only the mode symmetric about the new origin contributes to the amidship bending moment. Let this be the mode

$$\phi_0(\xi) = 1.$$

Hence the bending moment may be obtained from (2.12) in the form

$$M(\tfrac{1}{2}l, t) = \left\{ a l^2 \int_{-\frac{1}{2}}^{0} \rho g b(\xi) \left[\cos 2\sigma\xi - \frac{\bar{F}_0}{\mu'_0 \omega_0^2}\right] (-\xi)\,\mathrm{d}\xi + a\,\frac{\omega_e^2 \bar{F}_0 \bar{M}_0}{\omega_0^2 \bar{\mu}'_0(\omega_0^2 - \omega_e^2)} \right\} \sin(\omega_e t - \sigma),$$

where we have defined $\sigma = \frac{1}{2}kl$ and

$$\bar{F}_0 = l \int_{-\frac{1}{2}}^{\frac{1}{2}} \rho g b(\xi) \cos 2\sigma\xi\,\mathrm{d}\xi,$$

$$\bar{M}_0 = l^2 \int_{-\frac{1}{2}}^{0} [\mu'(\xi)\,\omega_0^2 - \rho g b(\xi)]\,(-\xi)\,\mathrm{d}\xi.$$

Now since the shear force at the end $\xi = \frac{1}{2}$ is zero, when the beam oscillates freely in the 0 mode at frequency ω_0, it is clear that

$$l \int_{-\frac{1}{2}}^{\frac{1}{2}} [\mu'(\xi)\,\omega_0^2 - \rho g b(\xi)]\,\mathrm{d}\xi = 0.$$

Hence

$$\mu'_0 \omega_0^2 = l \int_{-\frac{1}{2}}^{\frac{1}{2}} \rho g b(\xi)\,\mathrm{d}\xi.$$

If we define
$$A_1 = \int_{-\frac{1}{2}}^{0} (1-4\alpha\xi^2)\cos 2\sigma\xi\,\xi\,\mathrm{d}\xi$$
$$= \frac{1}{4}\left(1+\frac{6\alpha}{\sigma^2}\right)\left(-\frac{\sin\sigma}{\sigma}+\frac{1-\cos\sigma}{\sigma^2}\right)+\frac{\alpha}{4}\left(\frac{\sin\sigma}{\sigma}+\frac{3\cos\sigma}{\sigma^2}\right),$$
and
$$A_2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} (1-4\alpha\xi^2)\cos 2\sigma\xi\,\mathrm{d}\xi$$
$$= (1-\alpha)\frac{\sin\sigma}{\sigma}+\frac{2\alpha}{\sigma^3}(\sin\sigma-\sigma\cos\sigma),$$
then we obtain
$$M(\tfrac{1}{2}l,t) = \rho g b_0 l^2 a\left\{\left[-A_1+\tfrac{1}{8}A_2\frac{\frac{1}{2}\alpha-1}{1-\frac{1}{3}\alpha}\right]+\tfrac{1}{8}A_2\frac{\frac{1}{6}\alpha}{1-\frac{1}{3}\alpha}\,\frac{\omega_e^2}{\omega_0^2-\omega_e^2}\right\}\sin(\omega_e t-\sigma).$$
This may be written more conveniently as
$$\frac{M(\frac{1}{2}l,t)}{\rho g b_0 l^2 a} = \left[f_0^{\mathrm{W}}(kl)+\frac{\omega_e^2}{\omega_0^2-\omega_e^2}f_0^{\mathrm{L}}(kl)\right]\sin(\omega_e t-\sigma),$$
where f_0^{W} and f_0^{L} have been defined accordingly.

Appendix C. Amidship wave-excited vibratory bending moment in mode 1 of a uniform beam

We wish to find an expression for $f_1(\sigma)$ given by
$$\rho g b l^2 a f_1(\sigma) = |M_1^{\mathrm{H}}(kl,\tfrac{1}{2}l,t)|,$$
where
$$M_1^{\mathrm{H}}(kl,\tfrac{1}{2}l,t) = a\bar{F}_1(kl,t)\frac{1}{l}\int_0^{\frac{1}{2}l}\phi_1(x_0)\,(\tfrac{1}{2}l-x_0)\,\mathrm{d}x_0$$
and $\sigma = \frac{1}{2}kl$. The integral may most simply be found using the first identity obtained in appendix A. For the uniform beam in mode 1 this becomes
$$\bar{M}_1(\tfrac{1}{2}l) = (\mu'\omega_1^2-\rho g b)\int_0^{\frac{1}{2}l}\phi_1(x_0)\,(\tfrac{1}{2}l-x_0)\,\mathrm{d}x_0 = EI\phi_1''(\tfrac{1}{2}l).$$
Hence
$$\int_0^{\frac{1}{2}l}\phi_1(x_0)\,(\tfrac{1}{2}l-x_0)\,\mathrm{d}x_0 = \phi_1''(\tfrac{1}{2}l)/\beta_1^4.$$

The quantities ϕ_1'' and $\bar{F}_1$ may conveniently be evaluated using a coordinate system with origin at the centre of the beam, as in appendix B. In terms of the non-dimensional coordinate ξ, the characteristic function for mode 1 is
$$\frac{\cosh 2\gamma\xi}{\cosh\gamma}+\frac{\cos 2\gamma\xi}{\cos\gamma}$$
where $\gamma = \frac{1}{2}\beta_1 l$. Hence
$$\frac{\phi_1''(\frac{1}{2}l)}{\beta_1^4} = \frac{l^2}{4\gamma^2}\left[\frac{1}{\cosh\gamma}-\frac{1}{\cos\gamma}\right]$$
and
$$\bar{F}_1(kl,t) = \rho g b\int_0^l \phi_1(x)\sin(\omega_e t-kx)\,\mathrm{d}x$$
$$= \rho g b l\int_{-\frac{1}{2}}^{\frac{1}{2}}\left[\frac{\cosh 2\gamma\xi}{\cosh\gamma}+\frac{\cos 2\gamma\xi}{\cos\gamma}\right]\cos 2\sigma\xi\,\mathrm{d}\xi\sin\omega_e t$$
$$= \rho g b l\left[\frac{2\sigma^2}{\sigma^4-\gamma^4}(\sigma\sin\sigma-\gamma\tan\gamma\cos\sigma)\right]\sin\omega_e t.$$

Combining these results we find

$$f_1(\sigma) = \frac{\sigma^2}{2\gamma^2(\sigma^4-\gamma^4)}\left(\frac{1}{\cosh\gamma}-\frac{1}{\cos\gamma}\right)(\sigma\sin\sigma-\gamma\tan\gamma\cos\sigma).$$

References

Aertssen, G. & de Lembre, R. 1971 A survey of vibration damping factors found from slamming experiments on four ships. *Trans. N.E. Cst Instn Engrs Shipbuilders* **87**, 83–86.

Apostolov, V. S. 1969 Effect of the form of the hull extremities on the magnitude of hydrodynamic forces causing overall vibration. *Trans. Leningrad Shipbuilding Inst.* no. 66. (*BSRA Translation* no. 3476).

Bell, A. O. & Taylor, K. V. 1968 Wave-excited hull vibration measurements on a 47,000 tons deadweight tanker. *Shipp. Wld* **161**, 412–420.

Bishop, R. E. D. 1971 On the strength of large ships in heavy seas. *S. Afr. Mech. Engr.* December, 338–353.

Bishop, R. E. D., Eatock Taylor, R. & Jackson, K. L. 1973 On the structural dynamics of ship hulls in waves. *R. Inst. Nav. Archit. Paper* W1 (1973).

Bishop, R. E. D. & Johnson, D. C. 1960 *The mechanics of vibration.* Cambridge University Press.

Cleary, W. A., Robertson, J. B. & Yagle, R. A. 1971 The results and significance of the strength studies on the Great Lakes bulk ore carrier 'Edward L. Ryerson'. S.N.A.M.E. Symposium on Hull Stresses in Bulk Carriers in the Great Lakes and Gulf of St Lawrence Wave Environment, July 1971, paper G.

Fukuda, J. 1966 Computer program results for ship behaviour in regular head waves. *Proc. 11th ITTC, Tokyo,* 433–436.

Gerritsma, J. & Beukelman, W. 1967 Analysis of the modified strip theory for the calculation of ship motions and wave bending moments. *Int. Shipbldg Prog.* **14**, 319–337.

Goodman, R. A. 1971 Wave-excited main hull vibration in large tankers and bulk carriers. *Naval Architect* (*J. R. Inst. Nav. Archit.*), no. 1, 167–184.

Grim, O. 1960 A method for a more precise computation of heaving and pitching motions in smooth water and in waves. *3rd Symp. on Naval Hydrodynamics, Scheveningen,* 483–518. Washington: U.S. Govt. Printing Office, (1962).

Hoffman, D. 1972 Analysis of ship structural loading in a seaway. *Marine Technol.* **9**, 173–194.

I.S.S.C. 1970 *Proc. Fourth Int. Ship Struct. Congr.* Tokyo.

I.T.T.C. 1969 *Proc. Twelfth Int. Towing Tank Conf.* Rome.

Joosen, W. P. A., Wahab, R. & Woortman, J. J. 1968 Vertical motions and bending moments in regular waves. A comparison between calculation and experiment. *Int. Shipbldg Prog.* **15**, 15–31.

Kaplan, P. 1969 Development of mathematical models for describing ship structural response in waves. *Rep. Ship Struct. Comm. natn. Res. Coun., Wash,* no. SSC-193.

Korvin-Kroukowsky, B. V. & Jacob, W. R. 1957 Pitching and heaving motions of a ship in regular waves. *Trans. Soc. nav. Archit. mar. Engrs, N.Y.* **65**, 590–632.

Little, R. S. & Lewis, E. V. 1971 A statistical study of wave-induced bending moments on large oceangoing tankers and bulk carriers. *Trans. Soc. nav. Archit. mar. Engrs, N.Y.* **79**, 117–156.

Miles, M. 1970 The theoretical statistical distribution of the peaks of combined springing and wave-induced stress loads. *Rep. Ship Lab. natn. Res. Coun. Canada,* no LTR-SH 103.

Moor, D. I. 1967 Longitudinal bending moments on models in head seas. *Trans. R. Inst. Nav. Archit.* **109**, 117–165.

Newman, J. N. 1970 Applications of slender body theory in ship hydrodynamics. *Ann. Rev. Fluid Mech.* **2**, 67–94.

Ogilvie, T. F. 1969 The development of ship-motion theory. *Rep. Dep. Naval Archit. Mar. Engng Univ. Michigan,* no. 021.

Thomson, W. T. 1965 *Vibration theory and applications.* London: George Allen and Unwin.

van Gunsteren, F. F. 1970 Springing, wave-induced ship vibrations. *Int. Shipbldg Prog.* **17**, 333–347.

Weinblum, G. & St Denis, M. 1950 On the motions of ships at sea. *Trans. Soc. nav. Archit. mar. Engrs, N.Y.* **58**, 184–248.

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